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# VII. *The Approximate Solution of Various Boundary Problems by Surface Integration* combined with Freehand Graphs.*

*By* L. F. RICHARDSON.

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§ 1. General.

§ 2. $(\partial^4/\partial x^4+2\,\partial^4/\partial x^2\partial y^2+\partial^4/\partial y^4)\gamma=0$ in the body; and on the boundary $\gamma$ and $\frac{\partial\gamma}{\partial n}$ given.

§ 2·1. General.

§ 2·2. Example—Stress in a stretched notched plate.

§1. The method of solving boundary problems now to be described, consists in drawing, at a guess, a rough graph of the lines along which the integral is constant, and testing its accuracy by perimeter integration applied (in the approximate form of summation) to circles (or other simple figures) enclosed within the given boundary, in such a way as to find the integral at the centre of each circle, in terms of the integral and its space rates round the circumference; then embodying the corrections so found in a new graph and repeating the process until improvement becomes slow. In doing this one begins with a circle as large as will go into the region, or if the region is long and narrow, with several such circles. This gives a first approximation to the integral in the centre of the region. The graph is then re-drawn, and circles of half the radius of the first are used to test the parts extending from the centre to the boundary. Then, after re-drawing, still smaller circles are used for testing nooks and corners. Next the largest size of circles is used again, and so on.

The above process is for equations such as $\nabla^2\phi=0$, $\nabla^4\phi=0$, but when we have on the right-hand side of the equations not zero but a known function of $\phi$ and of the independents, then a certain volume integral has to be taken as well as the perimeter integrals. This can be calculated with sufficient accuracy by adding up a number of arithmetical values. When, as commonly, these are the values of a function of two functions of position, then the work is made easier by drawing a graph of the contours of one of the functions on tracing paper and laying it over a graph of the contours of the other.

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* The particular kind of surface integration referred to is that which gives the value, at a single point inside the surface, of the integral of an equation which holds throughout the volume inside the surface. When, by symmetry, it is only necessary to apply it to a line lying in the surface it will be called "perimeter integration," *not* contour integration as the word "contour" is reserved to mean a line along which a function of position is constant.

The drawing ensures that the function shall satisfy the right boundary conditions, shall have any necessary discontinuities and shall elsewhere be "smooth." The testing by perimeter and volume integration and consequent adjustment ensures that it shall satisfy the particular differential equation obtaining throughout the body. The accuracy of the result may be improved, in the case of certain differential equations, by converting it into a table of numbers and then making successive arithmetical approximations.* Further, whenever it has been decided to attack a problem by this arithmetical method, it will generally be quickest first to improve the preliminary assumption by graphs and perimeter integration.

Errors in graphs of integrals of $\nabla^2\phi=0$ are easily seen because the shape of the chequers into which the contours of $\phi$ and the orthogonal curves divide the field is a known function of their position and direction †. When, however, we have to deal with rather more complicated equations such as $\nabla^2\phi=$constant, $\nabla^2\phi=k^2\phi$ or $\nabla^4\phi=0$, the shape of the chequers is no longer a known function of their position and direction, but, instead, certain relations have to be satisfied by the differences of the ratio of length to breadth of adjoining chequers, relations which are usually too subtle for the eye to perceive. Instead of attempting to perceive them one may use contour integration in the manner described. The same may also, of course, be applied to $\nabla^2\phi=0$.

An account of Green's functions connected with equation $\nabla^{2m}\phi=0$, where $m$ is an integer, will be found in two Papers by T. Boggio‡. He shows, among other things, that

$$0=\int(u\nabla^{2m}v-v\nabla^{2m}u)d\,(\text{volume})$$
$$-\sum_1^m{}_i\int\left\{\nabla^{2i-2}u\frac{d\nabla^{2m-2i}v}{dn}-\nabla^{2m-2i}v\frac{d\nabla^{2i-2}u}{dn}\right\}d\,(\text{surface}),\quad(1)$$

where $dn$ is an element of the outwardly directed normal.

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* L. F. Richardson, "Phil. Trans." A (1910).

† This statement applies only when $\phi$ is a function of two co-ordinates, these co-ordinates being of a certain common kind; *see* L. F. Richardson, "Phil. Mag.," February, 1908.

‡ Accademia Reale delle Scienze di Torino, 1900. Rendiconti del Circolo Matematico di Palermo, 1905.

§ 2·1. Let us consider more particularly the equation

$$\nabla_1^4\gamma=\{\partial^4/\partial x^4+2\,\partial^4/\partial x^2\partial y^2+\partial^4/\partial y^4\}\gamma=0, \quad . \quad . \quad . \quad (2)$$

which is the one satisfied by the stress function in plain strain,* by the stream function in a viscous liquid moving under certain conditions, and by the displacement in a plane plate bent by forces at its edge.

Putting $m=2$ the formula (1) becomes

$$\int(u\nabla^4 v-v\nabla^4 u)d\,(\text{volume})$$

$$=\int\left(u\frac{d\nabla^2 v}{dn}-\nabla^2 v\frac{du}{dn}+\nabla^2 u\frac{dv}{dn}-v\frac{d\nabla^2 u}{dn}\right)d\,(\text{surface}). \qquad (3)$$

Then, if $\nabla^4 v=0$ everywhere except at one point P, where it becomes infinite in such a way that the integral of it over any volume, including this point, is $+1$, then

$$\int u\nabla^4 v d\ (\text{volume})=\text{the value of } u \text{ at P.}$$

If also $v$ and $\frac{\partial v}{\partial n}$ vanish on the boundary then

$$u \text{ at P}=\int\left(u\frac{d\nabla^2 v}{dn}-\frac{du}{dn}\nabla^2 v\right)d\,(\text{surface})+\int v\nabla^4 u d\,(\text{volume}). \ (4)$$

Here $u$ is supposed to be the integral which we require and $v$ a known function which is used to find $u$.

It is assumed that we are dealing with cases in which $\nabla^4 u$ either vanishes or is a given function of $u$ and of the co-ordinates, so that the volume integral can easily be estimated by adding up values at a number of points.

The special case which we chiefly require is that of a circle with P at the centre. Let the radius be $a$, then

$$v=\frac{1}{8\pi}\left[r^2\log\frac{r}{a}+\tfrac{1}{2}(a^2-r^2)\right]\dagger. \quad . \quad . \quad . \quad . \quad (5)$$

Forming $\nabla^2 v$ and $\frac{\partial}{\partial r}\nabla^2 v$ at $r=a$ it is found that, if $\nabla^4\gamma=0$, then

$$\gamma \text{ at centre of circle}=(\text{mean value of } \gamma \text{ on circumference})$$

$$-(\text{half radius})\times\left(\text{mean value of } \frac{\partial\gamma}{\partial r} \text{ on circumference}\right). \quad (5\text{A})$$

To determine these mean values on a graph one divides the circumference into a number of parts and adds up the values at their middle points arithmetically.

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* Love, "Theory of Elasticity," edition 1906, p. 201.

† Love, "Elasticity," 1906 edition, § 314.

Another special case which may be required is that where the boundary is an infinite straight line $y=0$ and P is at $x=0$, $y=a$. Then

$$v=\frac{1}{16\pi}[x^2+(y+a)^2][\log\{x^2+(y+a)^2\}-\log\{x^2+(y-a)^2\}]\ ^*$$
$$+\text{an arbitrary linear function}\ :$$

it follows that $\gamma$ at $x=0$, $y=a$ is equal to

$$\int_{-\infty}^{+\infty}\left[(\gamma)_{y=0}\ \frac{2a^3}{\pi(x^2+a^2)^2}-\left(\frac{\partial\gamma}{\partial n}\right)_{y=0}\ \frac{a^2}{\pi(x^2+a^2)}\right]dx.$$

The forms of $v$ are known for many more general shapes of boundary and with P anywhere. Thus O. Venske (Göttingen Nachrichten, 1891) gives $v$ for the circle, circular ring, space between intersecting straight lines, parallel straight lines and the space bounded by radii and concentric circles. But these will seldom be required for the graphic method.

§ 2·2. *Example.*—A thin elastic plate of the shape shown in Fig. 1 is stretched by forces applied uniformly over its ends. Required to make a graph of the stress function. The plate is 24 units long, 10 wide, and the semicircular notches have a radius of 2·1 and are placed centrally †. If $x$ and $y$ are co-ordinates in the plane of the plate then the stresses are $X_x=\frac{\partial^2\gamma}{\partial y^2}$, $Y_y=\frac{\partial^2\gamma}{\partial x^2}$, $X_y=Y_x=-\frac{\partial^2\gamma}{\partial x\partial y}$, where $\gamma$ is the stress-function satisfying $(\partial^4/\partial x^4+2\,\partial^4/\partial x^2\partial y^2+\partial^4/\partial y^4)\gamma=0$ ‡. Effect of the thickness of the plate is here neglected. The $y$ axis is taken parallel to the length of the plate. Over the ends $Y_y$=constant, therefore, $\frac{\partial^2\gamma}{\partial x^2}$=constant. Now, arbitrary values may be given to $\gamma$, $\frac{\partial\gamma}{\partial x}$ and $\frac{\partial\gamma}{\partial y}$ at any one point without affecting the stresses. So let $\gamma$, $\frac{\partial\gamma}{\partial x}$, $\frac{\partial\gamma}{\partial y}$ vanish at the centre of one end. Then, over this end $\gamma=Ax^2$, where A is a constant and $x$ is measured from the centre of the end. Further, $X_y$ vanishes over the end, and, therefore, by integrating $X_y=-\frac{\partial^2\gamma}{\partial x\partial y}$, it follows that $\frac{\partial\gamma}{\partial y}$ vanishes also. Now, it may be shown § that when the boundary stresses and the external forces, such as gravity acting on the internal matter, vanish, then $\frac{\partial\gamma}{\partial x}$ and $\frac{\partial\gamma}{\partial y}$ are constant round the boundary. These conditions are satisfied in the present case except over the ends of the plate. Consequently, if we imagine a length, $\gamma$, set up normal to the plane $x$, $y$ so as to form a surface, then on the longer edges of the strip and round the curves of the neck this surface is the plane which touches the surface $\gamma=Ax^2$ at the extremity of the end. The distribution of $\gamma$ is obviously symmetrical about the lines through the centre of the plate parallel to the straight sides and ends. Take the origin at the centre. The following is a record of the actual process by which the results to be given at the end were obtained.

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* Love, "Elasticity," 1906 edition, § 314.

† This problem is taken from a note by G. H. Gulliver, "Nature," April 14, 1910, Figs. 5 and 6.

‡ Michell, "Proc." Lond. Math. Soc., xxi. Love, "Theory of Elasticity," ed. 1906.

§ "Phil. Trans.," A Vol., 210, pp. 330—331.

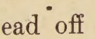
Short pieces of the contours of the surface $\gamma=Ax^2$ were first marked on the ends of the boundary of the drawing, and over the rest of the boundary were marked the ends of the contours of the plane which has $\gamma=A(5)^2$ at the straight parts of the longer edges and which has $\frac{\partial\gamma}{\partial x}=2A.5$ ; $\frac{\partial\gamma}{\partial y}=0$. In the drawing A was taken to be 2/5. Consequently the equal and opposite forces acting on the ends of the bar, being each equal to

$$\int_{x=-5}^{x=+5}\frac{\partial^2\gamma}{\partial x^2}\,dx, \qquad \text{are} \qquad \Big[4x/5\Big]_{-5}^{+5}=8.$$

This completed the boundary conditions. Next, the ends of the contours were joined across by lines drawn at a guess in Fig. 1A. The value of $\gamma$ at the centre of the plate was unknown, so that the dotted line $\gamma=1$ was particularly uncertain. The only places at which $\gamma$ vanishes are the points marked 0. Now, the large circle was drawn and divided into equal segments by dots. The values of $\gamma$ at the centres of these segments, starting from the left, were read off, thus—

0·9, 5·0, 9·0, 9.0, 4·0, 1·5, average 4·90.

In reading off values of $\frac{\partial\gamma}{\partial r}$ it is a help to have a piece of tracing paper with a line ruled on it, which is placed as tangent to the circle. The values of $\gamma$ at two dots a half unit on either



side of the tangent are read off and differenced. In other places it is more convenient to measure the distance along a radius between two contours, and to use for $\frac{\partial\gamma}{\partial r}$ the reciprocal of this distance multiplied by the difference in $\gamma$ between the said contours. Again, at or very near the boundary, $\frac{\partial\gamma}{\partial x}$ and $\frac{\partial\gamma}{\partial y}$ are given, and it is then often most accurate to take the direction cosines of the radius from a book of tables and so calculate $\frac{\partial\gamma}{\partial r}$. By these methods the following values of $\frac{\partial\gamma}{\partial r}$ were quickly found at the centres of the segments, reading from the left

$$0{\cdot}15,\ 2{\cdot}00,\ 3{\cdot}8,\ 3{\cdot}8,\ 2{\cdot}35,\ \text{zero, mean } 2{\cdot}01.$$

Therefore, by equation (5),

$$\gamma \text{ at centre}=4{\cdot}9-\frac{5}{2}\times 2{\cdot}01=-0{\cdot}12,$$

instead of about $+0{\cdot}6$, as indicated directly by the figure. Accordingly a new drawing was made in the quarter of Fig. 1 marked B, and in this $\gamma$ was made negative in the centre of the plate. It hardly seemed necessary to alter $\gamma=2, 4, 6, 8, 10$. The mean values of $\gamma$ and $\frac{\partial\gamma}{\partial r}$ were thereby changed so as to make $\gamma=-0{\cdot}5$ at the centre of the large circle, and this was also the value given directly by the graph as near as one could tell.

Next a circle with centre at the origin and radius 2·9 was taken and divided into segments as before. At the centres of these segments

$$\gamma=-0{\cdot}5,\ +0{\cdot}8,\ +1{\cdot}7,\ \text{mean } +0{\cdot}67,$$

$$\frac{\partial\gamma}{\partial r}=+1{\cdot}9,\ +0{\cdot}95,\ +0{\cdot}15,\ \text{mean } 1{\cdot}00.$$

Therefore, by equation (5),

$$\gamma \text{ at centre of the plate}=+0{\cdot}67-\frac{2{\cdot}9}{2}\times 1{\cdot}00=-0{\cdot}78.$$

Next a circle with radius 2·5 and centre at $y=4$, $x=2{\cdot}5$ was taken. The value of $\gamma$ at the centre of this, as given directly by the figure is 1·6. On the circumference, in counterclockwise order from the right, were found

$$\gamma=2{\cdot}7,\ 7{\cdot}0,\ 9{\cdot}5,\ 9{\cdot}4,\ 7{\cdot}0,\ 3{\cdot}6,\ 1{\cdot}2,-0{\cdot}1,\ -0{\cdot}5,\ -0{\cdot}7,\ -0{\cdot}4,\ +0{\cdot}5,\ \text{mean } 3{\cdot}27,$$

$$\frac{\partial\gamma}{\partial r}=0{\cdot}9,\ 2{\cdot}9,\ 3{\cdot}85,\ 3{\cdot}85,\ 2{\cdot}5,\ 0{\cdot}85,-0{\cdot}15,\ -0{\cdot}4,\ -0{\cdot}1,\ -0{\cdot}1,\ -0{\cdot}4-0{\cdot}4,\ \text{mean } 1{\cdot}11\,;$$

whence, by equation (5), $\gamma$ at the centre $=3{\cdot}27-\frac{2{\cdot}5}{2}\times 1{\cdot}11=1{\cdot}88$ instead of 1·6, as found directly. The graph was adjusted to

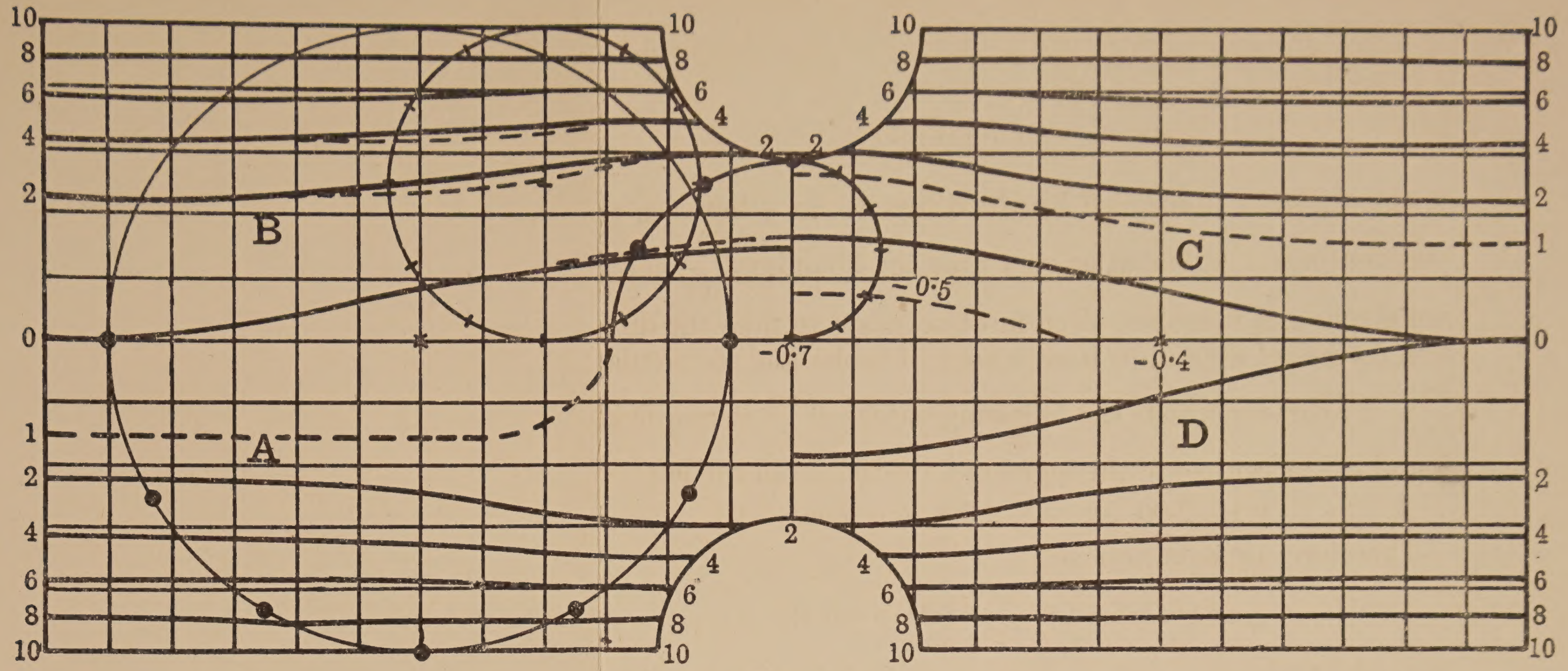


Fig. 1.

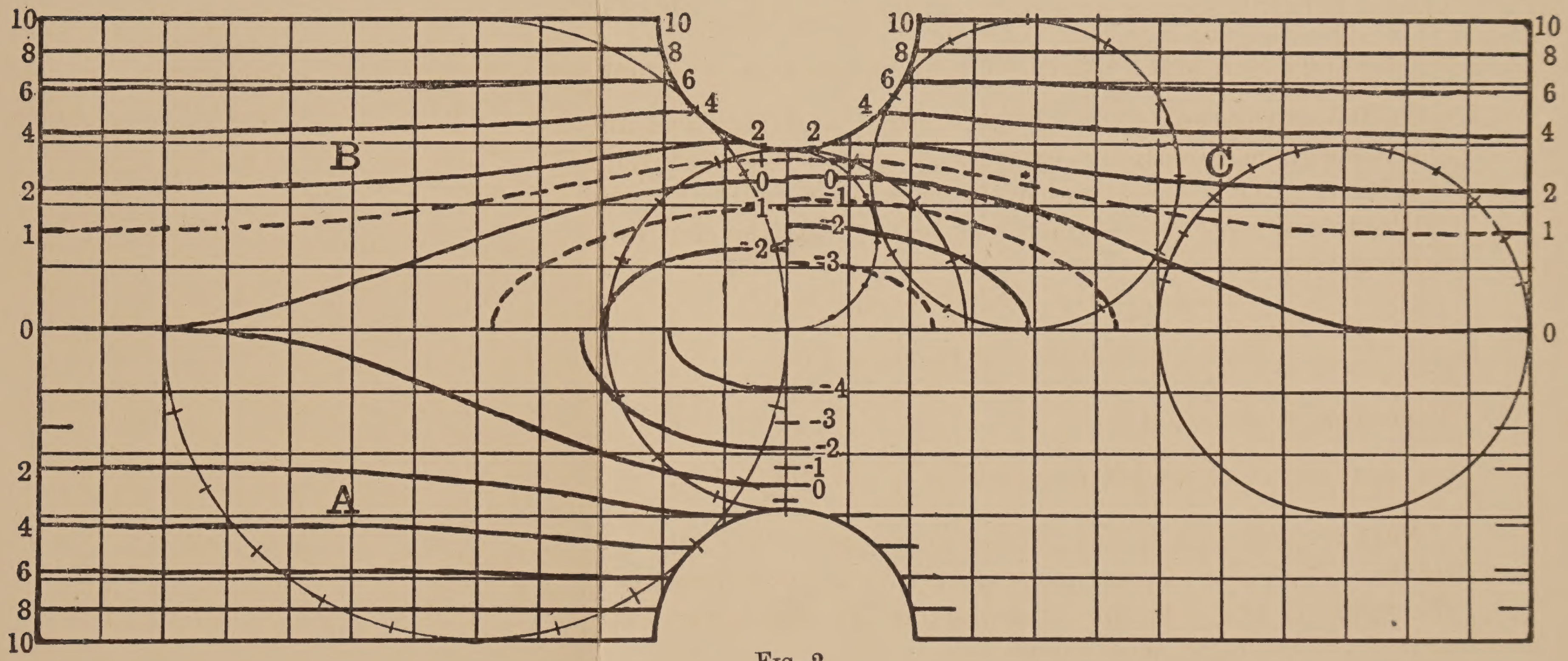


Fig. 2.

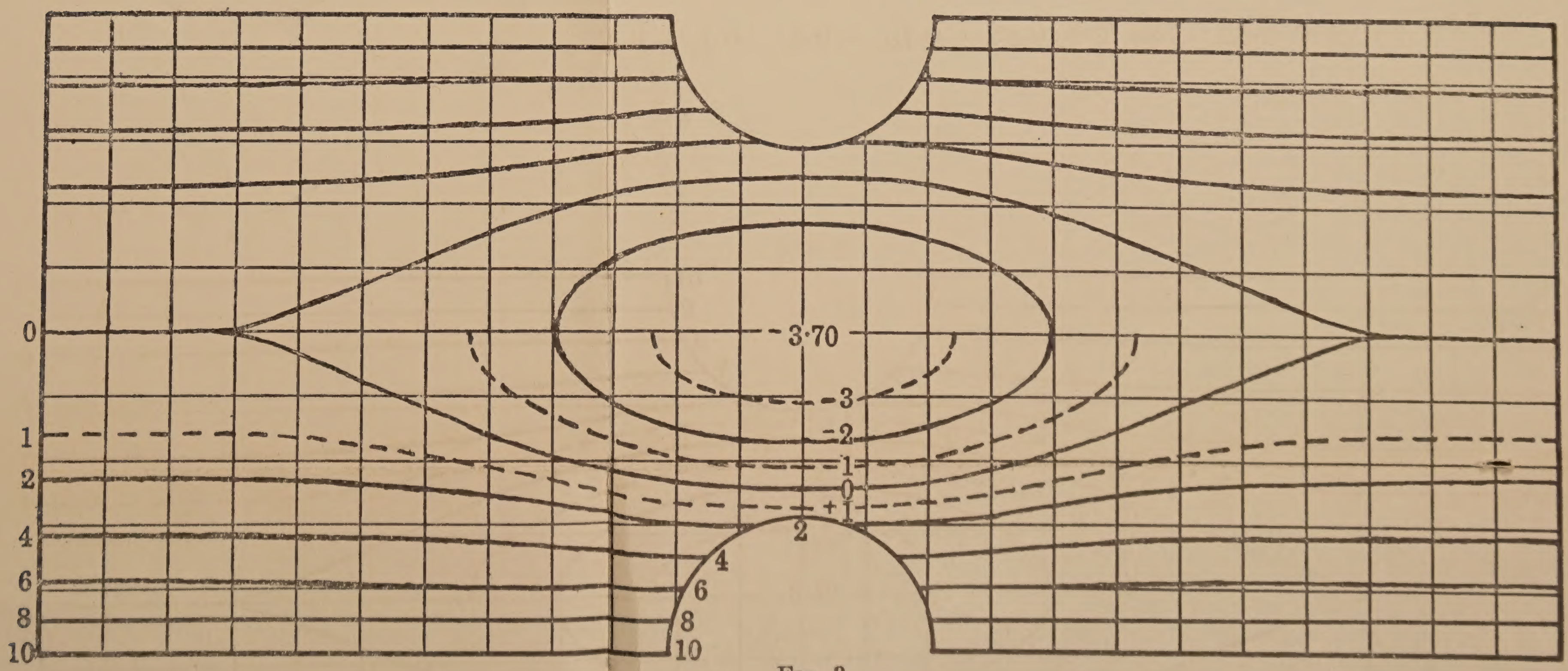


Fig. 3.

To face p. 80]

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accord with this value by moving the contour $\gamma=2$ into the position of the dotted line, so that $\gamma$ was increased by 0·30. Then, to preserve the smoothness of the function $\gamma$, the contour $\gamma=4$ had to be given a similar but smaller displacement. These displacements reduced the mean value of $\frac{\partial\gamma}{\partial r}$ on the circumference by only about 0·01. They increased the mean value of $\gamma$ by about 0·02. The value at the centre, by perimeter integration, was therefore changed by $0{\cdot}02+\frac{r}{2}(0{\cdot}01)=0{\cdot}03$. So it is seen that the slight displacement of the contour passing nearest to the centre of the circle, with accompanying changes to preserve smoothness, made much less change in the value of $\gamma$ found by perimeter integration than in the value read off directly, the ratio being as 0·03 to 0·30. The motion of the contours was stable. This property, when it exists, allows improvements to be made rapidly. As a result of the last change we have now, at $x=2{\cdot}5$, $y=4$, $\gamma=1{\cdot}90$ directly and $\gamma=1{\cdot}91$ by perimeter integration. Next, if the contour $\gamma=0$ be displaced into the dotted position the effects on the circle which we have just considered will be slight, the change in $\gamma$ compensating that in $\frac{\partial\gamma}{\partial r}$. This was done. The value at the origin by perimeter integration became $-0{\cdot}72$.

The graph now appeared to be fairly satisfactory. Some further tests were proposed to confirm it, and had they been applied they would doubtless have revealed what was now noticed—namely, that the stress $Y_y$ across the neck of the plate is probably fairly uniform. If quite uniform, then $\gamma$ is there of the second degree in $x$. But we have the values of $\gamma$ and $\frac{\partial\gamma}{\partial x}$ at the boundary $x=2{\cdot}9$, and so find at the centre of the plate $\gamma=-4{\cdot}75$ if $Y_y$ is uniform, instead of $-0{\cdot}7$ given by perimeter integration. It was at once clear that either $Y_y$ was not uniform or else the figure was seriously out. To decide which, the latest graph was copied into the quarter C of Fig. 1. An auxilliary diagram, Fig. 4, showing $\gamma$ as a function of $x$ along $y=0$ was used to find the points where $\gamma=1$ and $-0{\cdot}5$, and the contours through these points were then marked with dots in Fig 1c.

A circle was taken with centre at $x=1{\cdot}45$, $y=0$ passing through the origin and touching the boundary. At its centre perimeter integration gave $-0{\cdot}32$, whereas the value on the graph is there $-0{\cdot}1$. Suppose that to correct this discrepancy the zero line were shifted into the new position shown in Fig. 1D. This shift would cause, on the circumference of the circle which we have just considered, an increase in $\frac{\partial\gamma}{\partial r}$ and a decrease in $\gamma$, and would, therefore, create a demand for a further displacement in the same direction. The motion of the contours would be unstable.

It was, therefore, necessary to make a large displacement, preferably one overshooting the position of stable equilibrium of the contours. A fresh start was accordingly made from the assumption of uniform stress $\mathrm{Y}_y$ across $y=0$. (*See* Fig. 2A.) Here the contours $\gamma=2$, 4, 6, 8 were simply copied from Fig. 1C. The others were guessed. Perimeter integrations were then made around two circles in the same way as in the examples above and with the following results :—

| Co-ordinates of centre. | | Radius. | Value at centre by perimeter integration. | Value at centre directly from the graph. |
|---|---|---|---|---|
| $x$ | $y$ | | | |
| 0 | 0 | 2·9 | −4·30 | −4·75 |
| 0 | 5 | 5·0 | −1·34 | −0·8 |

These two discrepancies can be simultaneously reduced by making $\gamma$ less negative at the centre of the plate. The amount of change required was not apparent, but for trial a value half way between that in Figs. 1C and 2A was taken—namely, $\frac{1}{2}(-4\cdot7-0\cdot7)=-2\cdot7$. The corresponding graph is Fig. 2B. Taking again the large circle with centre at $x=0$, $y=5$ and $r=5$, it was found on its circumference that the changes from Fig. 2A increase the mean of $\gamma$ by 0·28 and the mean of $\frac{\partial\gamma}{\partial r}$ by −0·01. Therefore, $\gamma$ at the centre should be increased from −1·34 to $-1\cdot34+0\cdot28+2\cdot5\times0\cdot01=-1\cdot04$. Actually it is −0·9. Next from the circle with centre at the origin and radius 2·9 it was found that $\gamma$ should be −3·13 instead of −2·7 as assumed. Both these tests on Fig. 2B point to the true value at the origin being more negative than −2·7. Accordingly −3·7 was next tried. Fig. 2C is the corresponding graph. In making it the curve marked 1C in Fig. 4 was first drawn to look suitable and was then used to determine the ends of the contours in Fig. 2C along the $y$ axis. The lines $\gamma=4$, 6, 8 are the same as in the preceding graph; $\gamma=2$ has been slightly straightened. Around the circumference of the large circle with centre $x=0, y=5$ and radius 5 the distribution in Fig. 2C is simply the average of those in Figs. 2A and 2B; so $\gamma$ at its centre was made equal to the average of the values found by perimeter integration from Figs. 2A and 2B—namely, 1·19. Next the following tests were applied :—

| Centre of circle. | | Radius. | Angle between values used in summations. | $\gamma$ at centre. | | |
|---|---|---|---|---|---|---|
| $x$ | $y$ | | | By perimeter integration. | Directly. | Diff. |
| 0 | 0 | 2·9 | 22°·5 | −3·71 | −3·70 | −0·01 |
| 1·45 | 0 | 1·45 | 30° | −2·46 | −2·40 | −0·06 |
| 2·5 | 3·85 | 2·5 | 30° | +1·35 | +1·40 | −0·05 |
| 0 | 9·0 | 3·0 | 30°<br>15° | +0·09<br>+0·035 | 0·00 | +0·09<br>+0·035 |

The differences are all less than 0·1, which is 0·73 per cent. of the total range of $\gamma$ in the figure.

Enough has now been done to illustrate the process. Fig. 4

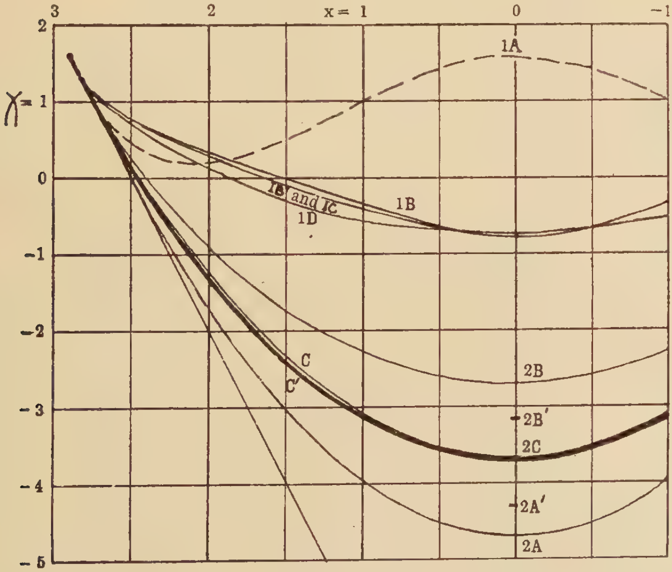

FIG. 4.

shows the successive stages at a glance. The final contours have been copied from Fig. 2C to Fig. 3, where they can be seen more clearly. In copying it was found desirable, in order to improve the smoothness, to shift the line $\gamma=0$ into the position shown dotted in Fig. 2c. The effect of this on the errors tabulated above will be less than 0·1 of $\gamma$.

From Fig. 3 the following relations between the stresses in the neck have been deduced. As they depend on second differences they are necessarily very inaccurate, and are to be regarded merely as suggestive. If a greater accuracy were required it would be best either to work with a larger diagram (Figs. 1, 2, 3 are reproductions of drawings 10 cm. × 24 cm.), or else to read from the graph a table of numbers and treat them by successive arithmetical approximations *. To return to the stresses in the neck. In the following table $\delta^2$ is the operator which takes the second difference of the tabulated numbers. Along $y=0$ :—

| | | | | |
|---|---|---|---|---|
| at $x=$ | $-1·45$ | $0$ | $+1·45$ | $+2·90$ |
| $\gamma=$ | $-2·46$ | $-3·70$ | $-2·46$ | $+1·60$ |
| Here the value $\gamma=-2·46$ was found by perimeter integration. *See* table on opposite page. | | | | |
| herefore $\delta^2\gamma=$ | ... | $+2·48$ | $+2·82$ | ... |
| herefore $\frac{\delta^2\gamma}{\delta x^2}=$ | ... | $+1·18$ | $+1·34$ | ... |

But the mean value of the stress $\frac{\partial^2\gamma}{\partial x^2}=Y_y$ is the total force 8. (*see* p. 79) divided by the width of the neck 5·9—that is, 1·355.

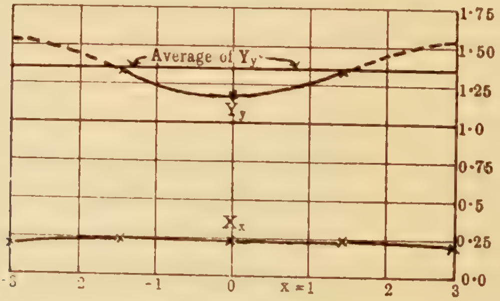

Fig. 5.

To give this mean value $Y_y$ must increase as the edge of the plate is approached. This is indicated in Fig. 5. The shear $Y_x$

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* L. F. Richardson, "Phil. Trans.", A., 1910, § 3·2.

vanishes by symmetry along $y=0$. The principal stress $X_x=\frac{\partial^2\gamma}{\partial y^2}$, at the origin, must be to $\frac{\partial^2\gamma}{\partial x^2}$ approximately inversely as the squares of the axes of the ellipse $\gamma=-3$ on Fig. 3. By measuring the ellipse its semi-axes were found to be 1·07 and 2·38. Therefore, $X_x=1{\cdot}18\times\left(\frac{1{\cdot}07}{2{\cdot}38}\right)^2=0{\cdot}24$ at the origin.

Again along $y=0$, since $\frac{\partial\gamma}{\partial y}=0$,

$$\frac{\partial^2\gamma}{\partial y^2}=(\text{curvature of line }\gamma=\text{const.})\times\frac{\partial\gamma}{\partial x}.$$

| | | |
|---|---|---|
| Hence at $x=$ | 1·45 | 1·90 |
| $X_x=\frac{\partial^2\gamma}{\partial y^2}=$ | 0·24 | 0·20 |

In conclusion, I have pleasure in thanking Prof. A. E. H. Love for references to Papers about $\nabla^4\gamma=0$ and Dr. C. Chree for reading and criticising the manuscript.

## ABSTRACT.

By boundary problems are here meant problems in which an unknown function of position—hereinafter called the integral—has to be found throughout a given region from the knowledge that it satisfies a given differential equation—hereinafter called the body equation—throughout the region, and that the integral and (if necessary) its space-rates are given on the bounding surface in such a way as to make it determinate throughout the region.

The method described applies to a wide range of body equations, and to any shape of boundary which can be represented satisfactorily by an outline drawn on paper. The discussion is for simplicity confined to integrals, which can be represented by a single graph showing a family of lines, along successive members of which the integral has successive constant values.

The method of solution depends on a knowledge of the value of the integral at a point inside some simple figure—say at the centre of a circle—as a function of the integral and (if necessary) its space-rates on the perimeter of the figure.

A drawing is made showing the boundary, and the contours are sketched in so as to satisfy the boundary equations, but elsewhere merely by guesswork. This graph is then tested by circles of various sizes and variously centred, and is adjusted accordingly. The testing and adjustment are repeated until sufficient accuracy is obtained or until improvement becomes too slow to be worth while. Some skill in freehand drawing is required.

As an example a rough determination is made of the stress-function in a notched plate, of special shape, when in tension.

## *The Electric Stress at which Ionisation Begins in Air.* By ALEXANDER RUSSELL, *M.A.*, *D.Sc.*, *M.I.E.E.*

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### 1.—*Introduction.*

Since the author read his Paper* on the electric strength of air in 1905 considerable additions have been made to our knowledge of the sparking voltages, in air at atmospheric pressure, between electrodes of various shapes. These results prove that when suitable precautions are taken this voltage has a perfectly definite value. It is now generally admitted that at atmospheric pressures the maximum electric stress is one of the main factors governing the discharge when the electrodes are at appreciable distances apart. The value of this quantity, however, when computed in the ordinary way by electrostatic theory, is found to vary with the size of the electrodes. It is much greater, for example, with small electrodes than for large electrodes, and, so far as the author is aware, no satisfactory explanation of this fact has been given.

In the preceding Paper it was shown that when equal spherical electrodes of 5 cm. diameter were used the values obtained for $R_{\max}$, the maximum value of the electric stress at the instant of the discharge, are approximately equal to 38 kv. per centimetre for sparking distances varying from 0·3 cm. to 10 cm. The values of $R_{\max.}$ for smaller distances are appreciably greater than 38, and for very minute distances $R_{\max.}$ attains a very high value. If E be the disruptive voltage, and if we suppose that $E-350$ is the voltage which is effective in producing the discharge, where 350 is the minimum sparking P.D. in air, we considerably increase the range over which $R_{\max.}$ is practically constant, and so we extend the practical usefulness of the mathematical formulæ. In what follows, however, we shall make no use of this assumption, but base our calculations, as in the postscript to the preceding Paper, on ordinary electrostatic theory. Before doing this it will be of interest to give an explanation of certain well-known experiments with electrodes at microscopic distances apart which have often been quoted to disprove the existence of a minimum sparking P.D.

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* "Phil. Mag." [6], Vol. II., p. 258, 1906.

### 2.—*The Minimum Sparking P.D.*

J. E. Almy* has given a direct experimental proof of the existence of a minimum sparking P.D. of about 350 volts. Those experimenters, therefore, who obtained sparking voltages smaller than this when the electrodes were only a micron or so apart must have mistaken the rush of current due to the actual contact of the electrodes for the rush of current which ensues when there is a spark.

Let us suppose that $x$ is the micrometer reading, and that the electrodes approach one another a distance $x-\xi$ owing to the electrostatic attraction when the P.D. between them is established. In the case of equal spherical electrodes, the author† has shown that when the ratio $\xi/a$ of the distance between them to the radius of either is of the order $1/10^4$, the electrostatic attraction can be represented by the expression $\lambda a E^2/8\xi$, with a maximum inaccuracy of less than three parts in 10,000, where $\lambda$ is the dielectric coefficient of the medium in which they are immersed and E is the P.D. between them. If we assume Hooke's law, the restoring force may be written $\alpha(x-\xi)$, where $\alpha$ is a constant. Hence the value of $\xi$ is given by

$$\lambda\frac{aE^2}{8\xi}-\alpha(x-\xi)=0.$$

The value of $\xi$, which is greater than $x/2$, corresponds to the stable position of equilibrium. As we increase the voltage, the roots of the quadratic become equal when $E^2=(2\alpha/\lambda a)x^2$. In this case the equilibrium is neutral, and the slightest increase in the value of E will cause the gap to close up, the electrostatic attraction overpowering the restoring force. Hence, if we plot out the values of E and $x$ for which the gap closes up, we get the line whose equation is

$$E=x\sqrt{\frac{2\alpha}{\lambda a}}.$$

For values of E, therefore, less than 350 the graph of E and $x$ will be a straight line, starting from the origin and making an angle $\tan^{-1}(2\alpha/\lambda a)^{1/2}$ with the axis of X. When, however, E equals 350 volts, a spark takes place between points on the electrodes which are at a certain distance, $\beta$, apart, which is, in general, greater than the minimum distance between the electrodes at this instant. Until the distance between the electrodes exceeds $\beta$, a spark takes place when the voltage is 350. Hence this part of the graph is a straight line parallel to the axis of X. For greater distances of the electrodes the value of E has to be increased in order to obtain a spark.

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* "Phil. Mag." [6], Vol. XVI., p. 456, 1908.

† "Proc." Roy. Soc., A., Vol. LXXXII., p. 524, 1909.

An examination of the experimental results* obtained for "sparking voltages" when the electrodes are at microscopic distances apart will show that they are in accordance with this theory.

### 3.—*Concentric Cylindrical Electrodes.*

In a valuable Paper on the electric strength of air, by J. B. Whitehead,† experiments on the voltage at which ionisation takes place at the surface of a wire placed along the axis of a metallic tube closed at the ends are described. Near each end of this cylindrical tube holes are drilled. Air enters by these holes and leaves through holes grouped together not far from the upper end. The circulation of the air is maintained by an exhaust fan. The air as it escapes passes over a suitable discharge terminal connected with a gold-leaf electroscope. The wire and tube are connected with the terminals of a transformer, so that a high alternating P.D. can be established between them.

In the experiments a low P.D. was first applied and then gradually increased. At a certain sharply defined voltage the leaves of the electroscope suddenly collapse. This is obviously due to the ionisation which accompanies the electrical breakdown of the air surrounding and close to the wire. Similarly, when the P.D. is gradually diminished from above the critical value, the ionisation suddenly becomes negligibly small when the voltage falls below this value.

Experiments are also described which show that the critical voltage is practically unaffected by the rate at which the air flows through the tube. The rate at which the ions reach the electroscope terminal is affected by this velocity, but for speeds varying from 0 to 370 ft. per minute the rate of discharge suddenly increases enormously at practically the same voltage. In the experiments referred to below, a velocity of about 150 ft. per minute was employed.

It was found that the critical voltage varied with the temperature. At 21°C. there was a fall of 0·22 per cent. in the critical voltage for a rise in temperature of one degree. This is in good agreement with Ryan's value of 0·27 per cent.[*] obtained by experiments on the voltage at which the corona first appears.

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* See P. E. Shaw, "Proc." Phys. Soc., Vol. XX., p. 289 and the references given there.

† "Proc." Amer. Inst. Elec. Eng., Vol. XXIX., p. 1,059, July, 1910.

The values of the maximum critical surface intensity $R_{\text{max.}}$ were computed by the well-known formula,

$$R_{\text{max.}}=E/\{a \log (b/a)\},$$

where E is the maximum P.D. between the wire and the tube, $a$ the radius of the wire and $b$ the inner radius of the tube. The values of $R_{\text{max.}}$ computed in this way from the experimental results were found to be independent of the metals of which the wire and tube were made. Varying the value of $b$ did not alter the value of $R_{\text{max.}}$, as we might have anticipated by the ordinary electrostatic theory. But the values of $R_{\text{max.}}$ varied largely with the radius $a$ of the wire. For instance, when the wire is 0·089 cm. in diameter, $R_{\text{max.}}$ is 77·1 kv. per centimetre, but when the diameter is 0·475 cm., $R_{\text{max.}}$ is only 51·4 kv./cm. It is important, therefore, to investigate the law which connects $R_{\text{max.}}$ with $a$.

*Temperature* 21°*C.* *Pressure* 76 *cm.*

| Diam. of wire in cm. | Metal. | Diam. of tube in cm. | Metal of tube. | $R_{\text{max.}}$ (observed) kv./cm. | $R_{\text{max.}}$ computed by (1). |
|---|---|---|---|---|---|
| 0·089 | Cu | 4·9 | Brass | 77·10 | 76·91 |
| 0·122 | ,, | 4·9 | ,, | 70·95 | 70·36 |
| 0·122 | ,, | 4·9 | ,, | 70·80 | |
| 0·156 | ,, | 4·9 | ,, | 65·88 | 65·93 |
| 0·156 | ,, | 4·9 | ,, | 65·88 | |
| 0·205 | ,, | 4·9 | ,, | 61·35 | 61·59 |
| 0·205 | ,, | 4·9 | ,, | 61·50 | |
| 0·205 | ,, | 6·35 | ,, | 62·08 | 61·59 |
| 0·205 | ,, | 6·35 | ,, | 61·78 | |
| 0·205 | ,, | 6·35 | ,, | 61·68 | |
| 0·254 | Al. | 6·35 | ,, | 58·75 | 58·59 |
| 0·276 | Cu. | 6·35 | ,, | 58·08 | 57·51 |
| 0·276 | ,, | 6·35 | ,, | 58·00 | |
| 0·276 | ,, | 6·35 | ,, | 57·74 | |
| 0·276 | ,, | 6·35 | ,, | 58·16 | |
| 0·276 | ,, | 6·35 | ,, | 58·08 | |
| 0·276 | ,, | 9·52 | Steel | 57·65 | 57·51 |
| 0·325 | Al. | 6·35 | Brass | 55·00 | 55·51 |
| 0·347 | Cu. | 6·35 | ,, | 54·50 | 54·75 |
| 0·3405 | ,, | 9·52 | Steel | 55·10 | 54·97 |
| 0·399 | Steel | 9·52 | ,, | 53·05 | 53·21 |
| 0·475 | ,, | 9·52 | ,, | 51·40 | 51·44 |

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[*] Russell, "Phil. Mag.," October, 1910.

### 4.—*Formula for the Maximum Electric Stress.*

The author has recently investigated a formula for the cooling of a wire maintained at a constant temperature in a draught of air.* The heat emissivity per square centimetre of the surface of the wire in this case is of the form $A+B/\sqrt{a}$, where A is the heat radiated directly and $B/\sqrt{a}$ is the heat carried away by the convection currents. This suggested trying whether a similar expression would give the value of $R_{max.}$ in Whitehead's experiments. It was found that the formula

$$R_{max.}=32+13{\cdot}4/\sqrt{a} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $R_{max}$ is in kilovolts per centimetre, gives all Whitehead's results within the limits of experimental error.

We conclude from the preceding table that the formula gives the values of $R_{max.}$ for all wires having diameters lying between 0·05 cm. and 0·5 cm., with a maximum inaccuracy of less than 1 per cent.

When the corona appears, substances affected by cathode rays, oxide of zinc for instance, become phosphorescent when placed near it. We may suppose, therefore, that electromagnetic waves capable of producing ionisation, as well as light waves, are being produced by the actions taking place in the corona. In the author's opinion this is due to collisions between positive and negative ions in the thin layer of the corona. The energy being radiated into space is the electrostatic energy of the ions which is set free when they collide and form a neutral system.

Let us suppose that $n$ is the concentration of the positive and negative ions in the corona. By the molecular theory of gases we know that the number of collisions per second between pairs of ions will equal $an^2$, where $a$ is a constant. The total number of collisions per second per unit length of the wire will be $2\pi a(an^2)$, and we may assume that violent ionisation begins when this attains a certain constant value. Hence, so far as the size of the inner wire is concerned, the critical value of $n$ varies inversely as $\sqrt{a}$. If we suppose that the value of the electric stress R at the surface of the corona is constant, and remember that the concentration $n$ of the ions in the corona is proportional to the potential gradient across it, we get

$$R_{max.}=R+\frac{B}{\sqrt{a}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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* "Trans." Amer. Inst. Elec. Eng., Vol. XXIII., 1904.

where R and B are constants. We have seen above that the experimental results can be exactly expressed by a formula of this nature.

### 5.—*Parallel Cylindrical Electrodes.*

In some of Steinmetz's experiments* two equal and slightly bent cylindrical rods were used as electrodes. To a first approximation, we may consider the electrostatic field between them as the same as that due to two parallel rods. It has to be carefully noticed that this case is very different from the case of a cylindrical rod and a coaxial tube. In the latter case we have a uniform corona all round the inner cylinder, but in the former the corona appears on the sides of the cylinders which face one another. The electric stress on the air at the surface of the cylinders is a maximum along the lines where the plane containing their axes cuts the inner sides of their surfaces. When they are close together this stress varies very rapidly from point to point along the circumference of a perpendicular section. It can readily be shown from the usual formulæ† that at any point, P, on the circumference of either cylinder we have

$$R=\frac{2q}{a}\cdot\frac{\sqrt{x^2+4ax}}{x+4a\sin^2(\theta/2)}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $q$ is the charge per unit length on the surfaces of the cylinders, $x$ the distance between them, $a$ the radius of either and $\theta$ the angle which the perpendicular from P on the axis of the cylinder makes with the plane which contains the axes of the cylinders. The value of $q$ is given by

$$q=\frac{E}{4\cosh^{-1}\{(x+2a)/2a\}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If R′ be the electric stress at the highest or lowest points of the cylinders, we have

$$\frac{R'}{R_{\text{max.}}}=\frac{x}{x+2a}; \quad \frac{R_{\text{min.}}}{R'}=\frac{x+2a}{x+4a}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It will be seen that, when $x$ is small, R′ is very small compared with $R_{\text{max.}}$ Hence it is only when $x$ is a large multiple of $a$ that the conditions approximate to those in Whitehead's experiments.

In one experiment Steinmetz used two cylindrical rods each

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* "Trans." Amer. Inst. Elec. Eng., Vol. XV., p. 281, 1898.

† Russell, "Alternating Currents," Vol. I., p. 102.

0·795 cm. in diameter, and the amplitude factor of the alternating P.D. was 1·42. Hence, computing $R_{max.}$ in the usual way,* we obtain the following table :—

*Two Cylindrical Electrodes.*
$2a = 0·795$; $R_{max.} = (1·42V/x)f$.

| $x$ in cm. | $f$. | V in effective kilovolts. | $R_{max.}$ in kv./cm. |
|---|---|---|---|
| 0·165 | 1·07 | 4·78 | 44·0 |
| 0·20 | 1·08 | 5·70 | 43·1 |
| 0·43 | 1·18 | 10·85 | 42·0 |
| 0·88 | 1·35 | 18·55 | 40·4 |
| 1·115 | 1·43 | 23·7 | 43·2 |
| 1·19 | 1·46 | 22·9 | 39·8 |
| 1·435 | 1·55 | 23·0 | 35·3 |
| 1·75 | 1·66 | 30·5 | 41·0 |
| 2·13 | 1·80 | 34·3 | 41·1 |
| 2·36 | 1·88 | 33·1 | 37·4 |
| 3·00 | 2·07 | 41·5 | 40·7 |
| 3·66 | 2·28 | 47·3 | 41·8 |
| 4·37 | 2·53 | 51·4 | 42·3 |
| 5·38 | 2·84 | 53·8 | 40·3 |
| 6·65 | 3·17 | 65·0 | 44·0 |
| 7·16 | 3·30 | 67·8 | 44·4 |
| 8·03 | 3·61 | 71·0 | 45·3 |
| 9·53 | 3·96 | 73·3 | 43·3 |
| 10·5 | 4·21 | 77·4 | 44·1 |
| 15·0 | 5·45 | 93·6 | 48·3 |

The mean of the values of $R_{max.}$ given above is 42·1. It will be seen that, although in order to get $R_{max.}$ we multiply the average potential gradient at the instant the voltage is a maximum by numbers varying from 1·07 to 5·45, yet the maximum departure from the mean is only about 14 per cent. If the values of V had been smoothed out by " curving " them in the usual way, the variation from 42·1 would only have been a few per cent. This proves that a knowledge of the mean value of $R_{max.}$ is of practical importance.

An examination of the above results shows that $R_{max}$ diminishes from the value 44 when $x$ is 0·165 cm. to a minimum value of about 39 when $x$ is 2·5, it then gradually increases. When $x$ is 15 cm. $R_{min.}/R_{max.} = 0·904$, and consequently the conditions approximate to those in Whitehead's experiments where the electric stress is uniform over the whole surface of the wire. By formula (1) we find that $R_{max.} = 47·0$, when $2a = 0·795$. This is in good agreement with the value 48·3 found above.

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* " Proc." Phys. Soc., Vol. XX., p. 73.

In a second experiment Steinmetz used 2·819 cm. rods.

*Two Cylindrical Electrodes.*
$2a=2\cdot819$ ; $R_{max.}=(1\cdot42V/x)f$.

| $x$ in cm. | $f$. | V in effective kilovolts. | $R_{max.}$ in kv./cm. |
|---|---|---|---|
| 0·24 | 1·03 | 5·9 | 36·2 |
| 0·28 | 1·03 | 7·0 | 37·1 |
| 0·44 | 1·05 | 9·94 | 33·9 |
| 0·48 | 1·06 | 11·6 | 36·6 |
| 0·81 | 1·09 | 19·3 | 36·9 |
| 1·03 | 1·11 | 21·6 | 32·9 |
| 1·25 | 1·14 | 27·0 | 35·0 |
| 2·26 | 1·25 | 41·0 | 32·2 |
| 3·33 | 1·37 | 52·0 | 30·4 |
| 3.78 | 1·41 | 60·5 | 32·0 |
| 4·52 | 1·48 | 66·2 | 30·8 |
| 4·55 | 1·49 | 68·4 | 31·8 |
| 5·23 | 1·57 | 74·0 | 31·5 |
| 5·56 | 1·61 | 79·0 | 32·4 |
| 6·02 | 1·65 | 84·6 | 32·9 |
| 6·45 | 1·69 | 93·0 | 34·6 |
| 7·37 | 1·77 | 98·5 | 33·6 |
| 8·10 | 1·82 | 103·7 | 33·5 |
| 9·45 | 1·97 | 118·4 | 35·1 |

The mean of the values of $R_{max}$ in the preceding table is 33·7. In this case, also, there are indications that $R_{max.}$ is great when the cylinders are close together, and that it passes through a minimum value as they are separated. This minimum value occurs when $x$ is about 4 cm. Undue importance, however, must not be attached to the above results, as the author has made no attempt in his calculations for $f$ to allow for the slight curvature of the axes of the cylindrical electrodes.

In the last experiment recorded in the above table $R_{min.}/R_{max.}$ is only 0·59. We would, therefore, expect that the value of $R_{max.}$ found by (1) would be different from 35·1. The value found by this formula is 43·3.

### 8.—*Spherical Electrodes. Kowalski and Rappel's Results.*

A very careful series of experiments on the sparking voltages between equal spherical electrodes has been recently published by Kowalski and Rappel.* It will be interesting, therefore, to take their results and examine the law according to which $R_{max.}$ varies with $x$ and $a$.

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* "Bull." Acad. Sci. Cracovie, Vol. V., p. 707, May, 1909. Translated in the "Phil. Mag.", Vol. XVIII., p. 699, November, 1909.

If we take their values for $R_{\max.}$ and $x$ for any set of experiments with given electrodes we find that they lie very approximately on a smooth curve. The curves for $R_{\max.}$ for spherical electrodes are roughly similar to the corresponding curves for cylindrical electrodes. When the electrodes are close together $R_{\max.}$ is great; as they are separated it attains a minimum value, and then seems to increase according to a linear law.

### 9.—*Formula Connecting Electric Stress and Sparking Distance.*

Trying a formula analogous to that for the cooling of a hot sphere in a liquid, the author finds that when $x/a=0{\cdot}1$ the formula

$$R_{\max.}=22+24/\sqrt{a} \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

agrees sufficiently well with Kowalski's results. When, however, $x/a=0{\cdot}2$, the formula becomes

$$R_{\max.}=23+17{\cdot}5\ \sqrt{a} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

and similar formulæ are obtained for other values of $x/a$.

The following empirical formula will be found sufficiently accurate for all values of $x/a$ between 0·1 and 2, and for all values of $a$ between 1 cm. and 15 cm., these being the limits for Kowalski's experiments:—

$$R_{\max.}=21+10\frac{x}{a}+\frac{11+1{\cdot}3(x/a)}{\sqrt{a}} \quad . \; . \; . \; . \; . \; . \quad (8)$$

It follows from this formula that $R_{\max.}$ has a minimum value when

$$x=0{\cdot}36a^{\frac{3}{4}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

For example, when $a=1$,

$$R_{\max.}=32+10x+1{\cdot}3/x \quad . \; . \; . \; . \; . \; . \quad (10)$$

| $x$ (cm.). | $R_{\max.}$ by (10). | $R_{\max.}$ (Kowalski). |
|---|---|---|
| 0·1 | 46·0 | 46·0 |
| 0·5 | 39·3 | 38·8 |
| 1·0 | 43·3 | 43·2 |
| 1·5 | 47·9 | 47·7 |

Similarly, when $a$ is 5, we have

$$R_{\max}=25{\cdot}9+2x+2{\cdot}9/x \quad . \; . \; . \; . \; . \quad (11)$$

| $x$ (cm.). | $R_{\max.}$ by (11). | $R_{\max.}$(Kowalski). |
|---|---|---|
| 0·5 | 32·7 | 32·8 |
| 1·0 | 30·8 | 30·9 |
| 1·5 | 30·8 | 30·8 |
| 1·8 | 31·2 | 31·2 |

### 10.—*Formula Connecting Voltage and Sparking Distance.*

For spheres,* when $x/a$ is not greater than 4, we have

$$R_{\text{max.}}=\frac{E}{x}f,$$

where $$f=1+\frac{x}{3a}+\frac{x^2}{45a^2}, \text{ approx.}$$

This formula for $f$ when the spheres are at potentials $+E/2$ and $-E/2$ can be proved by the Euler-Maclaurin sum formula. It is identical with the corresponding formula proved by Schuster*† for the case when the spheres are at potentials E and 0, but its range is much more extended, as the error in our case is less than 1 per cent. even when $x$ is $4a$. Hence, with the same limitations as in formula (8), we have

$$E=\frac{x\,R_{\text{max.}}}{1+x/3a+x^2/45a^2}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

### 11.—*Comparison of the Maximum Electric Stress with the Electrostatic Attraction at the Instant of the Discharge.*

It is of interest to compare the maximum electric stress at the instant of the discharge with the electrostatic attraction between the electrodes at the same instant. It would be suggestive, for instance, if the minimum value of $R_{\text{max.}}$ occurred when the electrostatic attraction between the electrodes was a maximum. Making the assumption that $R_{\text{max.}}$ is constant for electrodes of given size, the author finds that the electrostatic attraction between them is a maximum when $x=1{\cdot}2a$ approximately. The first of the following tables indicates that when $x$ has this value the electrostatic attraction is a maximum.

*Kowalski's Experiments with 2 cm. Spheres.*

| $x$ (cm.). | $R_{\text{max.}}$ (Kowalski). | Attraction in dynes. | $x$ (cm.). | $R_{\text{max.}}$ (Kowalski). | Attraction in dynes. |
|---|---|---|---|---|---|
| 0·085 | 47·6 | 237 | 0·672 | 39·8 | 625 |
| 0·136 | 42·9 | 297 | 0·875 | 41·9 | 690 |
| 0·177 | 40·9 | 325 | 0·953 | 42·6 | 705 |
| 0·220 | 39·9 | 367 | 1·07 | 44·0 | 726 |
| 0·316 | 38·5 | 435 | 1·35 | 46·4 | 721 |
| 0·372 | 38·5 | 478 | 1·53 | 47·9 | 708 |
| 0·494 | 38·7 | 543 | 1·75 | 49·0 | 650 |
| 0·537 | 38·8 | 563 | 2·22 | 51·3 | 594 |

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* "Phil. Mag." [6], Vol. II., p. 258, 1906.

† "Phil. Mag." February, 1890.

The above table shows that $R_{max.}$ has a minimum value when $x$ equals 0·34, and indicates that the attraction has its maximum value, not when $x$ has this value, but when $x$ is 1·2 approximately.

*Kowalski's Experiments with* 3·85 *cm. Spheres.*

| $x$ (cm.). | $R_{max.}$ (Kowalski). | Attraction in dynes. | $x$ (cm.). | $R_{max.}$ (Kowalski). | Attraction. in dynes. |
|---|---|---|---|---|---|
| 0·103 | 44·2 | 507 | 0·843 | 34·9 | 1,570 |
| 0·145 | 40·9 | 584 | 0·929 | 35·1 | 1,650 |
| 0·181 | 39·2 | 653 | 1·04 | 35·4 | 1,740 |
| 0·238 | 37·7 | 765 | 1·25 | 36·0 | 1,910 |
| 0·311 | 36·3 | 887 | 1·37 | 36·8 | 1,980 |
| 0·387 | 35·7 | 891 | 1·47 | 37·4 | 2,050 |
| 0·507 | 34·9 | 1,180 | 1·66 | 38·2 | 2,140 |
| 0·690 | 34·6 | 1,400 | ... | ... | ... |

In this case $R_{max.}$ has its least value when $x$ equals 0·7 cm. approximately; but when $x$ is 1.66, *i.e.*, when $x/a$ is 0·86, the electrostatic attraction is still increasing.

The Author is at present engaged in examining by means of simplified formulæ sparking voltages, obtained with unequal spherical electrodes. He therefore postpones making any general deduction from the above results.

## ABSTRACT.

In the "Proceedings" of the American Institution of Electrical Engineers for July, 1910, Prof. J. B. Whitehead publishes the values of the electric stress at which ionization begins in air. The electrodes he used in his experiments consisted of a metal tube and a cylindrical wire coaxial with it. Alternating pressures were employed, and the inner wires used had diameters varying in size from 0·089 to 0·475 cm. If $a$ be the radius of the inner wire, the Author has found that the expression $32 + 13{\cdot}4/\sqrt{a}$ gives all Whitehead's experimental results for the maximum electric stress in kilovolts per centimetre with a maximum inaccuracy of less than 1 per cent. The experiments show that the electric stress at which ionization occurs is independent of the metals used for the electrodes and of the inner radius of the outer tube. It depends merely on the radius of the inner wire. It is shown that Steinmetz's experimental results on the sparking distances between parallel rods are in substantial agreement with Whitehead's figures. Possible reasons for the discrepancies are given. An empirical formula is given which is based on experimental results recently published by Kowalski and Rappel for the sparking voltages between equal spherical electrodes. The results indicate that the electric stress at the moment of discharge has a minimum value when the distance between the electrodes is a certain function of their radius. The author has neglected the currents of electrified air which stream round the electrodes before the discharge takes place. These currents probably modify appreciably

the values obtained for the disruptive stress at the moment of discharge. The striking similarity between the formula for the heat emitted per unit area at the surface of a hot wire cooling in air and the formula for the electric stress at the surface of an electrified wire at which ionization first begins is pointed out.

## DISCUSSION.

Dr. C. Chree expressed his interest in the Paper, and remarked that when the electrodes were close together electrostatic attraction might produce an actual transfer of matter from one electrode to the other. The effect of the electrostatic attraction on the diminution of the distance between the electrodes would depend upon the nature of the apparatus used.

Prof. Strutt remarked that the effects which the Author had discussed in his Paper might be attributed to causes similar to those suggested by Dr. Chree.

Dr. Erskine-Murray gave some particulars of Dr. Shaw's experiments on the subject, and asked the Author if he would extend his researches to electrodeless discharges.

# IX. *A Demonstration of the Phase Difference between the Primary and Secondary Currents of a Transformer by Means of Simple Apparatus.* *By* F. T. Trouton, *Sc.D., F.R.S., Professor of Physics, University College, London.*

Received January 27, 1910. Read January 27, 1910.

A simple arrangement, such as described below, serves well for demonstrating Maxwell's relation for the difference in phase between the primary and secondary currents in a coreless transformer, as well as several other interesting points, both with a core and without one.

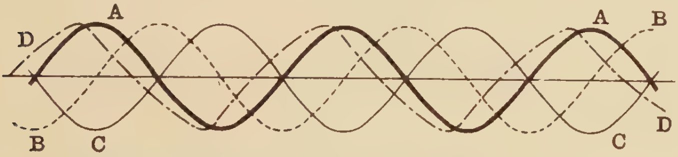
The essential part consists of a little inductive motor of primitive type. Two small horse-shoe electromagnets of slightly different length are placed standing vertically and across each other in such a way that their respective lines of poles are at right angles and in addition are in a horizontal plane. Over these, on a pivot, is placed a copper disc; this can be made to rotate like Arago's disc by the rotary field which is obtained by the passage through the magnets of the primary and secondary currents from a coreless transformer.

The difference in phase of the primary and secondary currents of such a transformer is given by the relation $\rho = \pi/2 + \tan^{-1} Lp/R$ when L and R are the inductance and resistance of the secondary circuit. By introducing additional inductances their ratio can be made large, so that their difference in phase approximates to 180 deg. and consequently we then find that the torque producing rotation of the disc becomes very small or practically vanishes. On the other hand, if the resistance is made relatively large, rotation is obtained; for then the phases of the two component fields being nearly at right angles are in the best condition to produce a rotating field.

If an iron core, consisting of fine wires, is introduced into this transformer the rate of rotation of the disc is generally much reduced; this is brought about by the effective or equivalent phase difference approaching nearer 180 deg., and is due to the hysteresis of the iron. Perhaps a graph giving the wave-forms makes this matter clearer. As only the phases are of interest for the purpose in hand no attempt is made in the diagram to give amplitudes to scale.

If we suppose, for simplicity, that an E.M.F. of simple harmonic type is applied to the primary, we then can take the impressed magnetic flux in the iron core to be also of simple harmonic type. Let this be represented by the full line. The E.M.F. in the secondary is shown by the dotted line and is $\pi/2$ in respect to phase behind the flux. The secondary current may lag behind this by any angle between 0 deg. and 90 deg., depending on the value of $\tan^{-1}(Lp/R)$. We see then that the lag of the secondary behind the flux may vary between 90 deg. and 180 deg. The major limit of the lag is exhibited by the thin line.

In the case of a coreless transformer these are also the limits for the lag of the secondary current behind the primary current, because then the impressed flux and current are identical as to phase; but if there is a core with hysteresis the primary current has a wave form somewhat as shown in the diagram by the unsymmetrical line, and, though no longer a simple harmonic curve, is seen on the whole to be in advance of the impressed flux. Thus we see that the difference in effective phase of the primary and secondary currents should be increased by the introduction of an iron core.



A=Impressed Flux. B=Secondary E.M.F.
C=Secondary Current. D=Primary Current with Hysteresis.

Again, it is obvious from the diagram that we should be able to get a difference in phase greater than 180 deg. This is easily realised in the apparatus described. All that is necessary to show it is to arrange, by increasing the induction in the secondary circuit, that the phase difference of the currents without a core is nearly 180 deg., when, on introducing a core, reversal of the rotation is at once obtained.

The apparatus, when in this state, is very suitable for demonstrating the rule that increase of inductance increases the lag while increase of resistance diminishes it. For reversal in direction is, of course, more easily adjudged than simply change in speed.

That the advance of the primary current, or what is the same thing, the lag of the flux carrying with it the secondary current depends upon the amount of hysteresis of the core can be demonstrated by substituting a steel core for the soft iron one. This lag is shown through increased speed of the disc in the reverse direction. By bringing the disc to rest by suitably changing the resistance in the secondary a measure can be obtained of the relative hysteresis of the materials employed for the core of the transformer. Unfortunately, it is necessary to have the material finely sub-divided as the Foucault currents in a solid core complicate the phenomenon, thus limiting its use as a hysteresis tester.

The part played by Foucault currents in the question of phase is best examined experimentally by exaggerating them. This is done by removing the core and substituting in its place a coil through which our exaggerated Foucault currents are to run, but now under control, for we can add resistance or inductance as desired, and there are besides no hysteresis effects to complicate matters.

It is shown theoretically below that the closing of such a circuit should increase the lag between the primary and secondary, and that under certain circumstances the phase difference can be actually greater than 180 deg., so that it should be possible to realise reversal of the direction of rotation without an iron core. Experiment has amply confirmed this.

When, in addition to the primary, there are two secondary windings on a coreless transformer, we have the following relations holding :—

$$L\frac{dC}{dt}+M_1\frac{dC_1}{dt}+M_2\frac{dC_2}{dt}+RC=E\cos pt,$$

$$L_1\frac{dC_1}{dt}+M_1\frac{dC}{dt}+m\frac{dC_2}{dt}+R_1C_1=0,$$

$$L_2\frac{dC_2}{dt}+M_2\frac{dC}{dt}+m\frac{dC_1}{dt}+R_2C_2=0,$$

when L, $L_1$, $L_2$, R, $R_1$, $R_2$, C, $C_1$, and $C_2$ are the inductances, resistances and currents in the three circuits while $M_1$ and $M_2$ are the mutual inductances of primary with each secondary and $m$ is that between the secondaries themselves.

To determine the phase difference between the primary current C and, say, $C_1$, one of the two secondary currents, we can write, adopting exponential notation :—

$$(L_1ip+R_1)C_1+mipC_2=—M_1ipC,$$
$$mipC_1+(L_2ip+R_2)C_2=—M_2ipC.$$

Eliminating $C_2$, we have

$$\{(R_2L_1+R_1L_2)ip+R_1R_2—(L_1L_2—m^2)p^2\}C_1=\{—M_1R_2ip+(M_1L_2—M_2m)p^2\}C.$$

This gives us for the phase difference between C and $C_1$

$$\rho=\tan^{-1}\frac{M_1R_2}{(M_1L_2—M_2m)p}+\tan^{-1}\frac{(R_2L_1+R_1L_2)p}{R_1R_2—(L_1L_2—m^2)p^2}.$$

It may be noticed in passing that the value of $\rho$ falls back into Maxwell's expression for the lag of the secondary behind the primary current, namely, $\pi/2+\tan^{-1}(L_1p/R_1)$ when $R_2$, the resistance of the subsidiary winding is made to grow indefinitely great.

If there is no magnetic leakage $M_1L_2$ would equal $M_2m$, provided that none of the inductance $L_2$ is other than that from the coil wound on the transformer. In which case the first term can never be far from $\pi/2$.

The value of the second term can be made equal to, greater or less than $\pi/2$ by varying the value of $L_1$, the inductance of the secondary circuit. In the experiment described above, $L_1$ necessarily includes a considerable inductance external to the transformer, namely, that of the rotatory apparatus, so that $L_1\times L_2$ is always greater than $m^2$, to which it would be equal were the external inductance absent. For the angle to be greater than $\pi/2$ it is only necessary to make $(L_1L_2—m^2)p^2$ greater than $R_1\times R_2$. In that case the total lag of the secondary current behind the primary current is greater than 180 deg., and we get reversal in the direction of the rotation.

The effect of varying other quantities in the denominator of the second term can be easily examined experimentally. Thus, increasing $R_2$ is found to produce direct rotation while increase the frequency ($p^2$) tends to produce reversed rotation, both of which, of course, are in agreement with the expression for the lag.

The hysteresis effect can be greatly modified by subjecting the iron of the core to permanent magnetic fields, which can be done by passing a continuous current through an additional coil wound on our transformer. Moderate fields increase the hysteresis effect so that the lag of the secondary current behind the primary is increased. This can be best observed by arranging first that the lag is almost 180 deg. when, on applying the continuous magnetic field reversal of rotation occurs. Large fields, on the other hand, give rotation in the direct sense.

The first is due to bringing the magnetic cycle arising from the alternating currents into the region represented by the knee of the hysteresis curve ; while the second effect is due to the core being permanently saturated, when it acts in respect to the relatively feeble alternating fields as a non-magnetic substance.

## DISCUSSION.

Dr. W. E. Sumpner remarked that the apparatus showed the effects of phase in a simple and striking way, and was likely to be useful for lecture purposes. But it was not altogether an advantage that so many phenomena can be shown by it. The phase difference between the two currents is due to the complex action of many causes. There is no simple connection between the phase of the primary current and that of the magnetism of the core of the transformer. Alterations in the secondary or tertiary circuit not only affect the relationship between the core magnetism and the secondary circuit, but also between the primary current and the magnetism of the core. This is true whether the transformer core is of air or of iron, but in the latter case, with enclosed iron cores the effects are most complex and do not lend themselves to simple explanations.

Prof. Silvanus P. Thompson shared Dr. Sumpner's doubt on certain points. The design of the apparatus took his mind back to one of the early meetings of the Physical Society in 1878 when Mr. Walter Baily showed an identical piece of apparatus for producing Arago's rotations by two electromagnets beneath a copper disc. They were supplied with an imitation of a two-phase alternating current, made by passing currents from a battery through a double revolving commutator. It was in fact the very first two-phase motor.

Prof. G. W. O. Howe pointed out that the magnetising winding, by means of which Prof. Trouton magnetises the iron core of the transformer, was really another secondary winding, and if the magnetising current was obtained from a few cells, this winding would be practically short-circuited. The effects produced might therefore be due to the presence of this winding, and not to the fact that the iron core was magnetised.

He considered it preferable to demonstrate the phase relations of the primary and secondary currents of a transformer by means of a double projection oscillograph, with one strip in the primary and the other in the secondary circuit. In the absence of an oscillograph, the present ingenious and easily made device would undoubtedly be of great value.

Mr. A. Campbell drew attention to an instrument recently described in the "Philosophical Magazine" which consisted of two wires carrying a small mirror, and crossing at right angles in a magnetic field at right angles to them both. The primary current of a transformer was sent through one wire and the secondary through the other when the spot of light described loops which showed phase difference and hysteresis effects.

The Author, in reply to Prof. Howe, said that the magnetising coil employed had a high resistance in order to avoid the effect he mentioned.

## X. *A Note on the Experimental Measurement of the High-Frequency Resistance of Wires. By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London.*

Read January 27, 1910.

In a Paper read to the Institution of Electrical Engineers in December, 1909, the author described an apparatus for the measurement of the high-frequency resistance of wires with which many additional experiments have been made lately in the Pender Electrical Laboratory, introducing one or two little improvements which have conduced to greater accuracy.* The apparatus as originally described consisted of a pair of glass tubes $T_1T_2$ having bends at the bottom full of mercury and closed by air-tight stoppers at the top and bottom (see Fig. 1). These tubes are connected by a lateral inverted syphon made of barometer tube having in it some paraffin oil with an air bubble ($b$) in the centre. The arrangement therefore forms a differential air thermometer and the air bubble serves to show whether the air pressure in both the large tubes is the same. The vertical tubes are about 30 in. in length and 1·5 in. in diameter. In each tube is placed one of two exactly equal wires (W) to be tested. The wire is supported from a copper rod passing air tight through a rubber stopper and the bottom end attached to another copper rod hangs in the mercury in the lower bend by means of which an electric current can be passed through the wires.

Means are provided by a condenser, inductance and discharger for generating electric oscillations of any desired frequency. Also means for passing a continuous current of any strength through either wire. By means of throw-over switches ($P_1$ $P'$) the high frequency or the continuous current can be passed through either wire in the tubes and measured by means of suitable high-frequency hot-wire ammeters, $A_1$ $A'$, as described in the Author's previous Paper (*loc. cit.*). Neglecting for the moment some necessary corrections, the experiment made consists in passing a continuous current through one wire and a high-frequency current through the other similar wire and adjusting them until the rate of production of heat in both cases is the same, when a steady condition is reached, as shown by the indicating bubble remaining stationary. If R is the steady resistance of the wire and R′ is its high frequency resistance, and if A and A′ are the equi-heating values of the continuous and high-frequency currents respectively, then for thermal equilibrium we have $A^2R = A'^2R'$, and therefor $R'/R = A^2/A'^2$.

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* *See* J. A. Fleming. "Some Quantitative Measurements in Connection with Radio-telegraphy." "Proceedings of the Institution of Electrical Engineers," London. Vol. XLIV., p. 349, 1910.

One essential condition of success is that the high-frequency current must be measured by an ammeter capable of giving

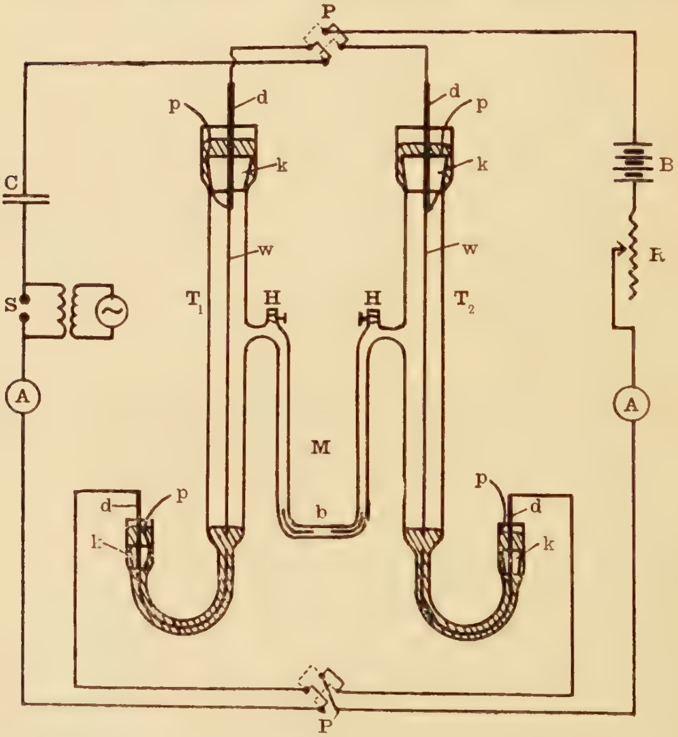

Fig. 1.

the true root-mean-square value. Such an ammeter has been described by the author in detail in the Paper referred to as read before the Institution of Electrical Engineers.*

There are, however, a number of precautions to be taken in

* This hot-wire thermo-electric high-frequency ammeter is now being manufactured by Mr. Robert Paul, of Newton-avenue Works, New Southgate, London, N.

using the apparatus. In the first place, although the wires tested and the tubes in which they are placed are as nearly alike as possible, the conditions of loss of heat are not absolutely the same. There may be a small difference in the emissivity of the wires due to difference in polish, also a difference in superficial area. Again the glass tubes may not absorb or radiate heat quite equally; and therefore the rate of loss of heat from the wires is not quite the same.

Supposing then that currents are passed through the two wires such that the bubble remains at rest it shows that the air-pressure in the two tubes is equal, and therefore that the air in the two tubes has the same mean temperature. If the experiment is continued until a steady state is reached the rate at which heat is being removed from each wire is proportional to some constant depending on the nature of its surface and some function of the mean temperature of air in the tube. Hence, if we pass a continuous current, $A_1$, through one wire and a high-frequency current, $A'$, through the other, and if $t_1$ is this mean temperature in both tubes when the bubble is at rest we shall have

$$\begin{aligned} A_1^2R &= e_1 f(t_1), \qquad \dots\dots\dots \quad (1) \\ A'^2R' &= e_2 f(t_1), \end{aligned}$$

where $e_1$ and $e_2$ are constants appertaining to each wire and $f(t_1)$ is some function of the mean temperature of the surrounding air.

Again, if we cause the high-frequency and the continuous current to change wires and vary the value of the continuous current until we again reach a steady state, at which the mean temperature of the air in the tubes is $t_2$, and if $A_2$ is the continuous current required to thermally equilibrate the same high-frequency current $A'$, we have

$$\begin{aligned} A_2^2R &= e_2 f(t_2), \qquad \dots\dots\dots \quad (2) \\ A'^2R' &= e_1 f(t_1). \end{aligned}$$

From (1) we have

$$\frac{R'}{R} = \frac{e_2}{e_1}\left(\frac{A_1}{A'}\right)^2 \qquad \dots\dots\dots \quad (3)$$

and from (2) we have

$$\frac{R'}{R} = \frac{e_1}{e_2}\left(\frac{A_2}{A'}\right)^2. \qquad \dots\dots\dots \quad (4)$$

Hence from (3) and (4) we have

$$\frac{R'}{R} = \left(\frac{\sqrt{A_1 A_2}}{A'}\right)^2. \qquad \dots\dots\dots \quad (5)$$

In other words, if we equilibrate the same high-frequency currents by continuous currents $A_1$ and $A_2$, changing wires in the two measurements, then the ratio of the high frequency to the steady resistance is equal to the ratio of the product of the two direct currents to the mean square value of the high-frequency current. In the former experiments we assumed that the arithmetic mean of the two continuous currents should be used, but it is clearly more correct to take the geometric mean. Against the former experiments on this subject the criticism was made that the calculated value of the high-frequency resistance with which the observed value should be compared should be the high-frequency resistance to damped oscillations which, as Dr. E. H. Barton has shown, is somewhat larger than the high-frequency resistance to persistent oscillations or undamped high-frequency currents.* This correction is a small one, though not absolutely negligible. Hence, in the above formula $R'$ must be understood to mean the resistance to such oscillations as are employed in the experiment. Also, it was asserted that the heating effect due to the high-frequency current was partly due to the charging current of the condenser used.

This last is, however, a quite negligible source of error. The condenser used was a battery of Leyden jars, which, when charged and discharged, gave trains of oscillations, each having about 30 to 50 oscillations or so, as proved by the decrement in that circuit. Hence the actual charging current of the condenser is not more than 3 per cent. or so of the oscillatory current at the most, and its mean-square value is not more than about 0·1 per cent. Moreover, the resistance on which the low-frequency charging current operates is the ordinary or steady resistance of the wire, whilst the resistance to the oscillatory current is the much greater high-frequency resistance. Hence, any correction under this heading is negligible in comparison with other sources of error.

The greatest practical difficulty is to secure a sufficiently steady high-frequency current. This was generated by employing a motor-driven alternator to give current to a large high-tension transformer raising the potential of an alternating current having a frequency of 50 to a potential of 10,000 or 20,000 volts. This voltage was used to charge one or more Leyden jars, which were discharged across a spark-gap, an air-blast on the gap being used to steady the discharge. The frequency of the oscillations so created was measured in each case by a cymometer, and the mean square value of the current by one of the Author's hot-wire thermo-electric ammeters.

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* *See* Dr. E. H. Barton "On the Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge." "Proc." Phys. Soc., London, 1899, Vol. XVI., p. 409, or "Phil. Mag.," 1899, Vol. XLVII., p. 433.

The voltage of the alternator could be regulated by a resistance in its field so as to alter the number of spark discharges per second. The wires inserted in the tubes for test had in all cases potential wires attached to their ends brought out separately through the stoppers. The purpose of these was to enable a measurement of the resistivity of the wires to be made *in situ* and at the temperature actually used. The experiments were conducted as follows: The two wires of identical materials were selected and cut the right length, and, having been cleaned, the diameters and lengths were carefully measured. They were then attached by solder and screw clips to the holders and inserted in the tubes. After the apparatus was rendered air-tight, continuous currents were passed through the wires and the current and volt drop down these wires measured with the potentiometer, the currents employed being of about the same value as those actually used in the thermal balance. These measurements gave the resistivity ($\rho$) of the wire. Thus, if A is the continuous current through the wire in amperes and $v$ the volt drop down the wire of length $l$ and section $s$, the resistivity is given by $\rho = sv/l\text{A}$. Generally speaking, a number of such measurements were made and a curve set out so as to be able to determine the proper value of the resistivity corresponding to any steady current used.

This having been done, the high-frequency current A′ was passed through one wire, say, in the left-hand tube, and a continuous current, $\text{A}_1$, passed through the right-hand wire and adjusted until the bubble in the inverted syphon remained at zero, after sufficient time had elapsed to allow the thermal state to become steady. The currents were then reversed, and the alternating passed through the right-hand wire and a continuous current, $\text{A}_2$, through the left-hand wire. The ratio of the product of $\text{A}_1$ and $\text{A}_2$ to $\text{A}'^2$ was then calculated, and this gives the ratio of the high frequency (R′) to the steady resistance (R) of the wire.

The wires tested in these experiments were circular sectioned straight wires of copper (No. 30, No. 16, No. 14 S.W.G.), tin, eureka and steel, and two spirals of No. 16 bare copper wire.

For each straight wire the value of the quantity $\sqrt{h}=\pi d\sqrt{\frac{n}{\rho}}$ was calculated, where $d$ is the diameter in centimetres, $\rho$ the resistivity in C.G.S. units corresponding to the mean value of the continuous current A used, and the frequency $n$ of the oscillatory current A′. It has been shown[*] that if $\sqrt{h}$ has a numerical value in the above units exceeding, say, 10, then for such wires the ratio R′/R is given by a formula due to Dr. A. Russell,[†] which is nearly equivalent to

$$\frac{R'}{R}=\frac{\sqrt{h}}{2}+\frac{1}{4}+\frac{3}{32\sqrt{h}}-\frac{1}{16h\sqrt{h}}.$$

For values of $\sqrt{h}$ in the neighbourhood of 10 and upwards the two first terms suffice.

On the other hand, if $\sqrt{h}$ is less than 10 or so, then the ratio R′/R must be obtained by the aid of a formula called the Ber and Bei formula, first given by Lord Kelvin and re-obtained and discussed by Dr. A. Russell.

In these calculations we have used Lord Kelvin's formula, because it has been reduced to numerical form for certain values of $\sqrt{h}$ or its equivalent by Dr. Magnus Maclean.[‡] The values of R′/R for the non-magnetic wires for the values of the frequency and current employed have been accordingly calculated in Table, on opposite page.

The correction to be applied for the higher resistance of the wires to damped rather than undamped resistance has not been made in the calculated value of R′/R because the decrement of the oscillations was not observed in every case, but this correction can at most increase the calculated value by about 1 per cent. or less. Dr. Barton has shown that the correction to be applied to Lord Rayleigh's formula for the ratio of R′/R, which is the same as the first term of Dr. Russell's formula is multiplication by a factor $s/\sqrt{s+k}$, where

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[*] *See* J. A. Fleming, "The Principles of Electric Wave Telegraphy and Telephony," 2nd edition, p. 112, 1910, for references to the literature of this subject.

[†] *See* Dr. A. Russell, "Proc." Phys. Soc., London, Vol. XXI., 1909; also "Phil. Mag.," Vol. XVII., p. 524, 1909, "On the Effective Resistance and Inductance of a Concentric Main."

[‡] *See* Lord Kelvin, Presidential Address, "Journal," Inst. Elec. Eng., 1889, Vol. XVIII., p. 35, and Appendix. Dr. Russell has shown that this Table requires recalculating, and some revised Tables have been published by Mr. H. G. Savidge, *see* "Proc." Phys. Soc., London, Vol. XXII., p. 105, 1910.

TABLE I.—*The Measurement of the High-Frequency Resistance of Certain Wires.*

| Conductor tested. | Value of $\sqrt{h} = \pi d \sqrt{\frac{n}{\rho}}$ | Frequency $n$. | R.M.S. value of the H.F. current in amps. $= A'$. | Value of the direct current in amperes which balances $A'$ thermally. | | | Ratio of currents $\frac{A}{A'}$ | Ratio of resistances $R'/R = (A/A')^2$. | | Remarks. |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Right hand tube $= A_1$. | Left hand tube $= A_2$. | Value of $A = \sqrt{A_1 A_2}$. | | Measured. | Calculated. | |
| Bare straight copper wire, No. 30 S.W.G. dia. = 0·03149 cm. | 2·415 | 1,080,000 | 1·75 | 1·79 | 2·48 | 2·11 | 1·21 | 1·45 | 1·45 | Calculated from Lord Kelvin's formula $\rho$ = 1,800 (measured). |
| Bare straight copper wire, No. 16 S.W.G. dia. = 0·1626 cm. | 12·77 | 1,080,000 | 5·80 | 13·1 | 17·0 | 14·92 | 2·57 | 6·62 | 6·64 | Calculated from Dr. A. Russell's formula $\rho = 1{,}730$ (measured). |
| Bare straight copper wire, No. 14 S.W.G. dia. = 0·198 cm. | 15·4 | 1,060,000 | 6·70 | 18·5 | 19·7 | 19·1 | 2·86 | 8·10 | 7·95 | Calculated from Dr. A. Russell's formula $\rho = 1{,}760$ (measured). |
| Bare straight tin wire pure tin, dia. = 0·0265 inch | 2·79 | 1,060,000 | 1·85 | 1·93 | 2·04 | 1·98 | 1·07 | 1·15 | 1·13 | Calculated from Lord Kelvin's formula $\rho = 17{,}000$ (measured). |
| Bare straight Eureka wire, No. 26 S.W.G. dia. = 0·019 inch | 0·623 | 1,060,000 | 1·045 | 1·00 | 1·08 | 1·04 | 0·995 | 0·99 | 1·00 | Calculated from Lord Kelvin's formula $\rho = 23{,}500$ (measured). |
| Steel wire, 1·0 % C. … dia. = 0·017 inch | $0·906\sqrt{\mu}$ | 1,060,000 | 0·530 | 1·36 | 1·37 | 1·363 | 2·575 | 6·63 | — | $\mu = 196$ (calculated).<br>$\rho = 23{,}500$ (measured). |
| Steel wire, 0·8 % C. … dia. = 0·0156 inch | $0·906\sqrt{\mu}$ | 1,060,000 | 0·530 | 1·35 | 1·38 | 1·363 | 2·575 | 6·63 | — | $\mu = 196$ (calculated).<br>$\rho = 19{,}800$ (measured). |
| Spiral of No. 16 bare copper wire, 135 turns turns per cm. = 2·65 | — | 545,000 | 3·90 | 10·8 | 11·05 | 10·91 | 2·800 | $R''/R = 7·82$ | $R''/R'$ 1·63 | value of $R'/R$ calculated for $n = 545{,}000$ by Russell formula 4·80. |
| Spiral of No. 16 bare copper wire, 135 turns turns per cm. = 2·65 | — | 750,000 | 3·90 | 9·8 | 10·15 | 10·0 | 2·517 | $R''/R = 6·54$ | $R''/R' = 1·2$ | value of $R'/R$ calculated for $n = 750{,}000$ by Russell formula = 5·54. |

$s=\sqrt{1+k^2}$, and $k=\delta/\pi$, where $\delta$ is the semi-period decrement of the oscillations. Experiments made on similar circuits to those used in the case of these measurements showed that $\delta$ might be taken to have some value of the order of 0·04, and therefore $k=\delta/\pi$ of the order of $\frac{1}{80}$. Hence, for such a case, $s\sqrt{s+k}=1{\cdot}007$, or the correction is under 1 per cent. Since the value of R′/R obtained by experiment cannot be repeated with differences much less than 1 per cent., it has not been considered necessary to apply the correction to the calculated values of R′/R, but it can be seen that, as far as the present experiments go, it would, if applied, make the agreement with the observed ratio in some cases rather better and in others rather worse than it is without it. In any case, the conclusion to be drawn is that this experimental method furnishes a value for the ratio which is in very fair agreement with that predetermined by theory in those cases in which the latter can give us a value.

This being so, we can have confidence in it in other instances, such as those of the magnetic metals in which an unknown quantity, viz., the permeability, enters into the value of $\sqrt{h}$. In this case $\sqrt{h}=\pi d\sqrt{\frac{\mu n}{\rho}}$.

Accordingly, for straight wires of the magnetic metals, we have approximately the equation

$$\frac{\mathrm{R}'}{\mathrm{R}}=\tfrac{1}{2}\pi d\sqrt{\frac{\mu n}{\rho}}+\tfrac{1}{4},$$

and this equation can be used to find the value of $\mu$ when $n$, $d$ and $\rho$ are known and R′/R is experimentally determined. Observations were accordingly made with two steel wires of different carbon content about 1 per cent. and 0·8 per cent., using the same methods. It was found that the experimental value of R′/R, corresponding to a frequency of $1{\cdot}06\times10^6$, came out in both cases the same, viz., 6·63. The diameters of these wires were respectively 0·017 in. and 0·0156 in., and the experimentally determined resistivities 23,500 and 19,800 C.G.S. units, hence $\pi d\sqrt{\frac{n}{\rho}}=0{\cdot}906$ in each case, and we have the value of the permeability $\mu$ from the equation

$$6{\cdot}63=\tfrac{1}{2}\times0{\cdot}906\sqrt{\mu}+\tfrac{1}{4}$$

or

$$\mu=196 \text{ for both wires.}$$

Since the radius of the cross-section of each wire is nearly 0·02 cm., and since the root-mean-square value of the current is about 1·3 amperes or 0·13 absolute units, it is clear that the root-mean-square value of the magnetic force H at the surface of the wire is about 13 C.G.S. units. The value of the permeability above found, viz., 196, is a kind of mean value corresponding to an intermittent oscillating magnetic force, and not to any steady value of the magnetic force.

If a magnetic force of 13 C.G.S. units was steadily applied the permeability corresponding to it would be about 900 or 1,000 for such a steel as is here used. Hence we cannot infer from the static magnetisation curve the permeability corresponding to a high frequency intermittent magnetising force any given R.M.S. value, since the actual permeability for such a magnetising force so applied appears to be always much less than for the equal steady force.

In the last place two experiments were made on the high-frequency resistance of spirals of copper wire. It is well known that a close spiral of wire of good conducting material has a larger high frequency resistance than the same wire stretched out straight. Hence, for such a spiral we have three resistances to consider, viz., its steady resistance R, its high frequency resistance R′ and its spiral resistance R″. The ratio of R″/R can be determined in the above-described apparatus as easily as that of a straight wire.

A spiral of copper wire, No. 16 S.W.G., was made having 135 turns and 2·65 turns per centimetre of length. This was tested at two frequencies, viz., $n$=545,000 and $n$=750,000. In the former case R″/R was found to be 7·82 and in the latter 6·54. From the known value of R′/R for this wire at these frequencies it was found that R″/R′ was for the lower frequency equal to 1·63 and for the higher 1·2. These values show that the ratio R″/R′ appears to diminish with increase of frequency.

The subject of the high-frequency resistance of spirals has already engaged the attention of several mathematicians and experimentalists. The formulæ given so far for predicting it for high frequency currents do not seem to be in good agreement with experimental results.

The latest investigation is that of Dr. J. W. Nicholson (*see* "Proc." Phys. Soc., London, Vol. XXII., p. 114, 1910, "On the Effective Resistance and Inductance of a Helical Coil"). He considers mathematically such a case as is here dealt with experimentally, namely, a circular-sectioned wire wound into a spiral of fairly close turns in one layer and traversed by a high frequency current.

If $a$ is the radius of the cylinder on which the spiral is wound and $r$ is the radius of the section of the wire, and $n$ is the number of turns in a length, $z$, of the spiral, then the angle $a$ of the spiral is defined by $\tan a = z/2\pi na = \beta/\pi D$, where $\beta$ is the distance from turn to turn and D is the diameter of the spiral. When $a$ is greater than $12r$, and the frequency greater than 4,900 $r^2$, Dr. Nicholson arrives at an expression for the effective resistance of the spiral per unit length of the wire which he gives in the form (*see* his equation (46), p. 128, *loc. cit.*)

$$R = \left(\frac{\mu p\sigma}{2\pi r^2}\right)^{\frac{1}{2}} \left\{1 + \left(\frac{\sigma}{8\pi\mu pr^2}\right)^{\frac{1}{2}} + \frac{3}{4}\frac{\sigma}{8\pi\mu pr^2}\right\}$$
$$- \frac{\sigma}{32\pi a^2} \cos^2 a(1 - 5\sin^2 a)\left\{1 + \left(\frac{\sigma}{8\pi\mu pr^2}\right)^{\frac{1}{2}} - \frac{9}{4}\left(\frac{\sigma}{8\pi\mu pr^2}\right)^{\frac{3}{2}}\right\}$$

where $\sigma$ is the resistivity of the wire, $\mu$ the permeability, $r$ the radius of section, and $p = 2\pi n$, where $n$ is the frequency. This expression can be simplified as follows: In place of his R put R″ since he uses R to denote the effective resistance per unit of length of the spiral wire, and I have here employed R to signify the steady resistance per unit of length. In place, therefore, of $\sigma/\pi r^2$, write R, since this is the steady or ohmic resistance per unit of length of the wire. Also take $\mu = 1$, as we are dealing with non-magnetic wires, and in place of $\sigma/8\pi pr^2$ write $1/4h$, which makes $\sqrt{h} = 2\pi r\sqrt{\frac{n}{\sigma}} = \pi d\sqrt{\frac{n}{\rho}}$, as above; also write $d$ for $2r$ and D for $2a$.

With these changes, Dr. Nicholson's formula becomes

$$\frac{R''}{R} = \left\{\frac{1}{2}\sqrt{h} + \frac{1}{4} + \frac{3}{32}\frac{1}{\sqrt{h}}\right\} - \frac{1}{32}\frac{d^2}{D^2}\cos^2 a(1 - 5\sin^2 a)$$
$$\left\{1 + \frac{1}{2\sqrt{h}} - \frac{9}{32}\frac{1}{h\sqrt{h}}\right\}.$$

It will be seen that the first three terms in large brackets are the approximate expression, according to Dr. A. Russell's formula, for the ratio of the resistance per unit length of the wire for high-frequency currents to its steady resistance per unit of length, the wire being straight. This ratio is what we have called R′/R. Hence Nicholson's formula becomes

$$\frac{R''}{R} = \frac{R'}{R} - \frac{1}{32}\left(\frac{d}{D}\right)^2 \cos^2\alpha(1 - 5\sin^2\alpha)\left\{1 + \frac{1}{2\sqrt{h}} - \frac{9}{32\,h}\,\frac{1}{\sqrt{h}}\right\}$$

when $d$ is the diameter of the wire and D of the spiral. If the angle $\alpha$ is small the term $\cos^2\alpha(1 - 5\sin^2\alpha)$ is nearly unity, and positive in sign, and if $h$ is large the three last terms in large brackets are also not far from unity and positive.

Hence the formula clearly shows that for a value of $\alpha$ less than 26° 35′ R″/R should be *less* than R′/R by a small quantity. Now, this is contradicted by experiment. R″/R is always *greater* than R′/R, and the ratio of R″/R is as shown above, a quantity which is 1·63 or 1·2 for the spiral in question for the two frequencies 545,000 and 750,000.

Also the formula shows that when $\alpha = \sin' 1/\sqrt{5} = 26° 35'$ R″/R=R′/R; and that when $\alpha$ is greater than 26° 35′ R″/R is greater than R′/R. The formula is, therefore, qualitatively wrong in certain cases and not in agreement with the results of experiment. The only case in which it agrees with observation as regards ratio of R″ to R′ is when the angle of the spiral is greater than 26° 35′, but this case has little or no practical importance, as the inductance coils of importance are mostly spirals of small angle.

Similar experimental measurements of the ratio R″/R′ made for copper spirals of the same kind by T. P. Black* in 1906 gave results of the same order as those obtained by me, viz., values between 1·2 to 1·89 for frequencies of $10^6$ or that order and spirals made of wire 0·15 and 0·3 cm. in diameter.

It can be proved by elementary reasoning, as given by the author in the Paper read to the Institution of Electrical Engineers that the spiralisation of a wire must increase its high-frequency resistance, and this is experimentally confirmed by the measurement here given and by all those of Black. There is no doubt that a fairly close spiral of many turns made of No. 16 copper wire has a resistance to high-frequency currents with frequency $\frac{1}{2}10^6$ to $10^6$, which is from 1·5 to 2 times that of the same wire stretched out straight to the same currents. Hence any formula which makes it less or only greater by a very small percentage must be wrong.

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* *See* T. P. Black, "Widerstand von Spulen für Schnelle Elekt Swingungen," "Annalen der Physik," Vol. XIX., p. 157, 1906.

A formula given by Sommerfield gives, as Black has shown, too large a value for R″/R, and another formula given by Cohen has only been tested with rather low-frequency currents.

There seems, therefore, still room for mathematical investigation to obtain for us a formula for the resistance of spiral wires to high-frequency currents, which shall be as much in accord with the results of observations, as are the formulæ given by Lord Kelvin and Dr. Russell for straight conductors. The experimental work for this Paper has been very carefully carried out by Mr. A. D. Peacock, superintended by Mr. G. B. Dyke and by me; and to them I am indebted for the experimental results here given.

## ABSTRACT.

The Author refers to a Paper read by him in December, 1909, before the Institution of Electrical Engineers on Quantitative Measurements in Connection with Radio-Telegraphy " ("Journal," Inst. Elec. Eng., Vol. XLIV., p. 349, 1910), in which he described an apparatus consisting of a differential air thermometer having tubular bulbs into which similar wires could be placed, and by means of which a comparison could be made of the high-frequency (H.F.) resistance R′ of a straight wire and its steady or ohmic resistance R. If two equal wires have passed through one, a steady current A and through the other a H.F. current $A_1$, then if these currents are adjusted until the rate of heat evolution in each case is the same we have $A^2R = A_1{}^2R'$. Certain precautions are described in the Paper for eliminating inequalities, but by means of correct reading H.F. ammeters as devised by the Author, the ratio of the resistances R′/R can be determined from the ratio of the mean square currents $A^2/A_1{}^2$.

The H.F. currents used were obtained by condenser discharges and the equiheating steady current determined by means of the differential thermometer arrangement having two equal wires in the two tubular bulbs. It is then shown that the results for straight wires agree very well with the formulæ given by Lord Kelvin and by Dr. A. Russell for the ratio R′/R for wires of different sizes and for different frequency. It is pointed out that the correction to be applied for damped oscillations as compared with persistent oscillations is at most 1 per cent. in the cases measured and that the correction for the heating effect of the condenser charging current is negligible. The case of spiral wires is then discussed. The resistance R″ of a spiral is greater than that of the same wire R′ stretched out straight. In the cases examined the ratio R″/R exceeds R′/R by about 50 to 80 per cent. A formula given by Dr. Nicholson for R″/R is then discussed, and it is shown that it does not agree with the results of observation. Experiments are also described on the H.F. resistance of wires of magnetic metals, and it is shown that in this case the observed value of R′/R can be used to determine the permeability for small H.F. magnetising forces.

## DISCUSSION.

Dr. Russell said the verification of an abstruse mathematical formula was a notable achievement. The suggested method of getting the high-frequency permeability of steel wires was ingenious, and would be of value at low flux densities, but at high flux densities the assumption of a constant value of $\mu$ on which the formula had been obtained was not permissible. When the pitch of the helix was small and the frequency was not very high, he thought that the formula obtained by Esau, "Annalen der Physik," Jan., 1911, would be of value. Dr. Nicholson had attacked the much more difficult problem of a helix of considerable pitch. He was not prepared to say that the formula for the high-frequency resistance obtained by Dr. Nicholson was erroneous. In a straight wire the flux density was greatest round the circumference, but in a helix of small pitch it was greatest near the axis of the helix. For moderate pitches it was therefore difficult to see how the current would distribute itself. On experimental grounds, however, Dr. Fleming's criticism was quite justified, and his results would do much to encourage mathematical research.

Mr. Duddell drew attention to the importance of being able to predict the high-frequency resistance of spiral wires on account of their numerous applications. His own experiments had only been made on insulated wires. He found that the insulation increased the resistance owing probably to dielectric losses in the insulating material.

Dr. Erskine-Murray remarked that unless we stick to the definition of H.F. resistance as resistance to a uniform H.F., A.C., we shall soon be at cross purposes, since the resistance to a damped intermittent current is not the same thing, though in Dr. Fleming's particular case it does not differ by more than 1 per cent. He was interested to hear that the sources of error which he pointed out in Dr. Fleming's previous Paper on the subject do not amount to more than 1 per cent. under the conditions of the experiments. Wireless telegraphists would be grateful to Dr. Fleming for providing them with a method which will enable them to make much needed experiments, and we might look forward to obtaining a really useful knowledge of the subject in the course of two or three years.

Major O'Meara thought the method well worth following up, and said he hoped it would be followed up at the Post Office.

Dr. W. E. Sumpner observed that it did not really matter much whether a particular formula for spiral wires was correct or not. What was really important was that the tests showed Dr. Fleming's apparatus gave for the H.F. resistance of straight wires results in very close accord with established formulæ. It might, therefore, be reasonably expected to give the correct resistance for wires of all kinds, whether spirally wound or not.

Prof. E. Wilson drew attention to the inductive action between the two circuits which, though negligible for straight wires, might not be so when one was a spiral.

Prof. C. H. Lees remarked that we must depend upon experiments for knowledge of the high-frequency resistance of wires, and not upon mathematics.

Dr. J. W. Nicholson said the formulæ given in his Paper, referred to by Prof. Fleming, were restricted to the case in which the pitch of winding was not small. Emphasis was laid upon this fact in the introduction to his Paper, in which it was stated that Cohen's formula appeared to be quite satisfactory for all cases in which the pitch is small. Cohen's experiments strongly tended to show this, although it is necessary to notice, as Prof. Fleming has done, that these experimental tests were confined to rather low frequencies. He stated that his Paper proposed to deal with the other extreme case in which the pitch was not small, and that a connection between his formulæ and those of Cohen could not be made without a consideration of the difficult case in which the pitch is moderately small. But as, in the final statement of his results, this emphasis was not repeated, the fault of the misunderstanding was perhaps to a great extent his own. The precise restriction is that the pitch shall be of the same order of magnitude as the radius of the cylinder of winding, and in this case, the correction for spirality must be a small quantity. Since the $\alpha$ of his formula tends to be near $\frac{\pi}{2}$, $1-5\sin^2\alpha$ will be negative for all the cases to which the formula applies, and the correction for spirality is therefore positive, and not negative, as Prof. Fleming deduced by writing $\alpha$ a small quantity. The proof that spirality increases the resistance, given by Prof. Fleming in the Paper read before the Institution of Electrical Engineers, is very elegant.

It seems on consideration of the numerical magnitudes of the quantities in Prof. Fleming's experiments, that Cohen's formula ought to meet the case approximately, and that these experiments would supply a useful test of that formula for high frequencies. Prof. Fleming does not appear to have tested the formula in this way as yet, but the conditions of the experiments are somewhat close to those assumed by Cohen. In reality they are those of the intermediate cases, but $\alpha$ is so small that they are much nearer Cohen's conditions than to Dr. Nicholson's.

The AUTHOR, in reply, said he was glad to find that Dr. Russell substantially agrees with him that there is room for further investigation in the formula for the high-frequency resistance of spiral wires. If Dr. Nicholson's formula is valid only for spirals of large pitch, *i.e.*, greater than about 30 deg., then his formula is qualitatively right, although it has yet to be shown that it is quantitatively correct. Nevertheless the only case of practical interest is that of a tolerably close spiral for which Nicholson's formula fails. The author omitted to note that one of the assumptions made by Dr. Nicholson was that the pitch of the spiral must be large. Hence part of the force of this criticism is removed.

## XI. *The Measurement of Energy Losses in Condensers Traverse, by High-Frequency Electric Oscillations. By* J. A. FLEMING, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London, and* G. B. DYKE, *B.Sc., A.M.Inst. Elec. Eng.*

READ JANUARY 27, 1910.

IN apparatus for radio-telegraphy on the spark system, the oscillations are created by a discharge of a condenser, and the efficiency of the transmitting apparatus will therefore be affected by any energy losses in the condensers used. It is sometimes assumed that these energy losses are negligible, and that the chief sources of energy dissipation in the transmitter are the spark resistance and the high-frequency resistance of the circuits. This, however, is not the case, and the object of the experiments here described was to put in practice a simple method of measuring energy losses in condensers traversed by high-frequency currents. These losses may be regarded as if they were due to a resistance in series with the condenser, the dielectric being supposed to have no resistance. That is to say, we may consider that the actual condenser used is equivalent, as far as energy losses are concerned, to a condenser of the same capacity, but with perfect non-dissipative dielectric, having in series with it a resistance of some kind in which the condenser current dissipates energy. We cannot however, consider this resistance to be a constant resistance. It is a function of the condenser current, since the energy losses are not simply proportional to the mean square value of the condenser current. The assumed resistance is more analogous to the internal resistance of a voltaic cell. We can, however, state it as having a particular value for a certain current, or rather for the mean of two currents not very different in value.

After several different methods had been tried, the following plan seemed to give reliable and consistent results. The condenser to be tested is joined up in series with a circuit consisting of a copper wire of known diameter and resistance wound on a square frame so that its high-frequency resistance can be calculated, and its inductance measured. This circuit forms a non-radiative or feebly radiative circuit, hence if free oscillations are set up in this circuit they are damped out by resistance and condenser losses only. In circuit is also inserted a hot wire ammeter of the kind described by one of us (Dr. Fleming) as suitable for measuring high-frequency currents, and the circuit is interrupted also by a pair of mercury cups so that various short-wire resistances can be inserted into the circuit to vary its resistance by a known amount without sensibly altering its inductance.* This condenser circuit, which will be called the secondary circuit, is placed in contiguity to another primary circuit in which highly-damped oscillations are set up by means of an impact discharger. The primary circuits consist of a square circuit, of one or two turns of wire of the same size as the secondary circuit having in series with it one or two Leyden jars and an impact discharger.

The impact discharger used is similar in general principle to one devised by W. Peukert, but modified by Prof. Fleming.† It consists of two round discs, $A_1$, $A_2$ (*see* Fig. 1) of polished steel turned extremely true, case-hardened and ground dead flat. One of these $A_2$ has a hole F in the centre, the other $A_1$ is fixed to a shaft, B, running true in ball bearings. This shaft is carried in a frame, to which the stationary disc is fixed by adjusting screws, C, C; the two discs are insulated from each other and are placed so that their surfaces are truly parallel and separated only by a quarter of a millimetre. The frame carrying the discs is placed in a glass vessel filled with paraffin oil, and the upper disc is revolved by an electric motor at a speed of 2,000 revs. per min. The moving disc is connected to one insulated terminal, D, by a rubbing contact, E, on the shaft, and the fixed disc is connected to another insulated terminal, D′. When the upper disc revolves at a high speed it flings the oil out between the discs and fresh oil is sucked in at a hole, F, in the lower disc. There is therefore a continual circulation of the oil between the plates, and by means of adjusting screws the lower disc is placed with its surface perfectly parallel with the under surface of the revolving disc. If, then, these discs are made the discharger in a condenser circuit, the discharge takes place perfectly uniformly over the whole surface of the two discs, that is to say, it does not take place continually at one spot. Moreover, it is a dead-beat discharge. The spark is damped out of existence instantly, and although the oil becomes carbonised it is continually being renewed between the discs, and the products of decomposition do not remain between the plates.

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* *See* "Some Quantitative Measurements in connection with Radiotelegraphy," by J. A. Fleming, "Journal of the Institution of Electrical Engineers," Vol. XLIV., 1910, page 352, for a description of this hot-wire ammeter.

† For a description of Peukert's discharger, *see* "The Electrician," Vol. LXIV., p. 550, Jan. 14, 1910.

If two or more such dischargers are joined up in series, we have a very efficient impact discharger, which will run for hours by the aid of a small electric motor without attention.

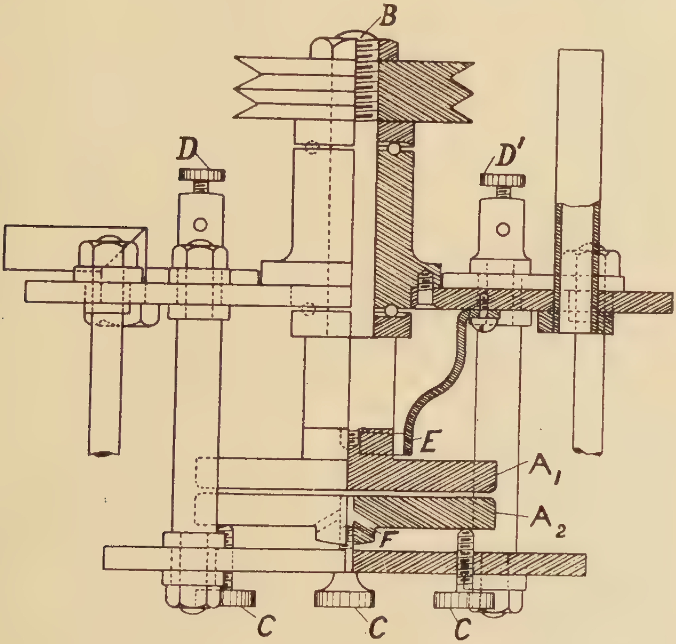

FIG. 1.

If, then, such a discharger is used in the primary circuit, the oscillations in that circuit are highly damped and practically the discharge is non-oscillatory. On the other hand, the effect on the secondary circuit is to set up in the latter its free vibrations. As there are an enormous number of sparks per second, we obtain very steady feebly damped oscillations in the secondary circuit.

Suppose, then, we measure the R.M.S. value of the current in the secondary circuit, we have four causes for the damping out of the free oscillations taking place in that circuit. These are as follows :—

(1) The high-frequency resistance R′ of the square circuit, which can be calculated from the dimensions of the wire and from the frequency, which last can be experimentally determined.

(2) The high-frequency resistance of the interpolated resistance $r$, which is also known.

(3) The resistance of the ammeter $r'$.

(4) The unknown source of energy loss in the condenser, which may be represented as due to a resistance $\rho$, which is not constant but is a function of the condenser current. If, then A is the R.M.S. value of this current in the secondary circuit as read on the ammeter, the total energy loss per second in that circuit is given by $A^2(R+r_1+r'+\rho)$, and if the inductance of the circuit is L and the frequency of the oscillations is $n$ then the total decrement $\triangle$ of the circuit is equal to $10^9(R+r_1+r'+\rho)/4nL$, where $\delta=10^9\rho/4nL$, is that part of the decrement contributed by the condenser.

If, then, we alter the resistance in the secondary circuit from $r_1$ to $r_2$ we have two corresponding observed values of the current, say $A_1$ and $A_2$. If an impact discharger is used in the primary, circuit there is no reaction between the secondary and primary because the primary spark ceases almost at once, and hence the moment the secondary oscillations begin the primary circuit is open. The impact discharger communicates at every discharge the same energy to the secondary circuit, and that energy is entirely dissipated as heat, hence we must have the equation

$$A^2(R'+r_1+r'+\rho)=\textit{a constant}=A^2R.$$

Accordingly, a curve whose ordinates are the total high-frequency resistance R and the mean square current $A^2$ is an equilateral hyperbola. If, however, the resistance $\rho$ is a function of the current, then if we slightly change the added resistance $r$ of the circuit from one value $r_1$ to another $r_2$, and take two corresponding readings of the current, we shall know the following equations :—

$$A_1^2(R'+r_1+r'+\rho)=A_2^2(R'+r_2+r'+\rho),$$

or

$$\rho=\frac{A_2^2r_2-A_1^2r_1-(A_1^2-A_2^2)(R'+r')}{A_1^2-A_2^2}.$$

The value of $\rho$ is considered to belong to the mean value of $A_1$ and $A_2$.

In this manner we have been able to measure for various forms of condenser the value of the equivalent resistance $\rho$ and of that part of the total decrement $\triangle$ contributed by the condenser. The first condenser used was a variable capacity oil condenser with semi-circular metal plates movable more or less in between each other by turning a handle. The oil used was a limpid vaseline oil. The results of the observations are set out in the various Tables in subsequent pages. In each case the equivalent resistance of the condenser $\rho$ is stated as corresponding to a certain current which is the mean of two observed currents, one slightly greater and the other slightly less than the mean value by which the resistance is calculated by the above formulæ. The results show that in all cases this equivalent resistance $\rho$ increases with the condenser current. The power W dissipated in the condenser is measured by the product of this resistance $\rho$ and the square of the corresponding mean current A, whilst the decrement $\delta$ is the quotient of $10^9\times$ this resistance by the quantity $4n\text{L}$, where $n$ is the frequency and L the total inductance of the circuit. In all cases the frequency of the oscillations $n$ for each experiment was measured by a cymometer.

In this manner we have tested an air condenser, a number of Leyden jars, a condenser made with ebonite as dielectric, and various forms of glass plate condenser, and a condenser made by Moscicki. The results are embodied in the following Tables, which will explain themselves. We have also measured the bulk of the dielectric in each case and stated the energy dissipation D in microwatts per cubic centimetre of the dielectric and also the R.M.S. value of the electric force E in the dielectric calculated from the charging current and the capacity and the geometric dimensions of the dielectric. From these figures it appears highly probable that the energy loss per cubic centimetre of the dielectric varies as a power of the electric force not far from 3 or 4.

It is evident from the Tables that the internal losses in the condensers increase with great rapidity with an increase in the potential difference between the plates, or rather with the electric force in the dielectric.

It is clear, from these observations, that even in the case of air condensers and oil condensers, energy losses are not entirely absent, whilst for certain forms of glass plate condenser this energy loss may amount to a very considerable value. It is of great importance to determine the law according to which these energy losses increase with the electric force in any dielectric so as to be able to predetermine from observations made with one charging current what the losses would be for any other charging current. The case is exactly analogous to that of the energy losses produced in iron by magnetisation. From the empirical law, discovered by Steinmetz, connecting together the energy loss per cycle of magnetisation with the maximum value of the flux density during the cycle, we can determine the energy loss due to magnetic hysteresis corresponding to any other maximum flux density and for any given material by means of a formulæ of the type, $H = \eta B^{1 \cdot 6}_{\text{max.}}$, where $\eta$ is constant for one particular kind of iron, and $B_{\text{max.}}$ is the maximum value of the flux density during a cycle of magnetisation.

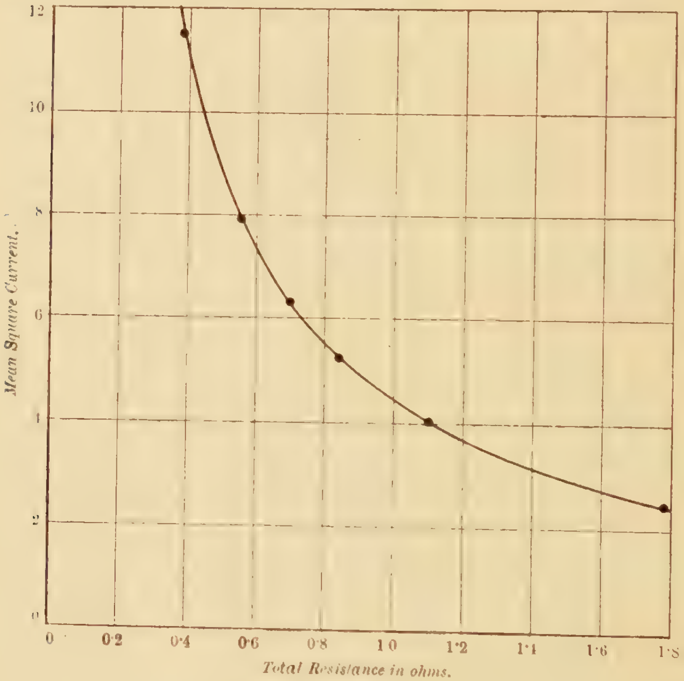

Fig. 2.

Researches conducted by Prof. R. Threlfall on the conversion of electric energy in dielectrics (*see* "Physical Review," Vol IV., p. 457; Vol. V., p. 21), have shown that the energy losses in dielectrics in the form of spheroids tested in a rotating electric field, and therefore at low frequencies, can be represented by a formula of the same type, viz., $W=aE^n$, where E is the internal electric force and W the energy dissipated and $a$ and $n$ are constants. Threlfall states that for most ordinary dielectrics the exponent $n$ lies between 1·5 and 1·95, but he found that the coefficient $a$ varies a good deal from sample to sample. Our experiments have shown that the energy loss in condensers traversed by high-frequency currents may be made to fit into a formula of a similar type within certain ranges of the electric force. There are, however, some cases in which the observed values can only be made to fit into such a formula by variation of the values of $a$ and $n$ over particular ranges of electric force.

The source of the energy loss in condensers inserted in circuits in which high-frequency currents are flowing, may, however, be complex. Perhaps they are partly due to true conduction losses, partly to brush discharges at the edges of the plates and partially to true dielectric losses which may be tentatively described as dielectric hysteresis.

TABLE I.

Condenser tested ............... Adjustable oil condenser.
Capacity ............................ 0·00266 microfarad.

| Total high-frequency resistance of circuit. R. | R.M.S. value of condenser current. A. | Power delivered to secondary circuit. $A^2R$. |
|---|---|---|
| 0·383 | 3·39 | 4·40 |
| 0·556 | 2·81 | 4·39 |
| 0·701 | 2·51 | 4·42 |
| 0·843 | 2·30 | 4·44 |
| 1·099 | 2·01 | 4·44 |
| 1·779 | 1·58 | 4·45 |

The first set of experiments were made with a condenser consisting of metal plates immersed in vaseline oil. By measurements of the current A and total resistance R of the circuit as given in Table I., a series of observed values of $A^2$ and R were found which fit extremely well on the hyperbola $A^2R=4{\cdot}42$.

This hyperbola is represented by the firm line in Fig. 2, and the dots on it represent the observations made in Table I.

The observations recorded in Table IA show that for this condenser the equivalent resistance is small and nearly constant over the range of electric force used.

TABLE IA.

| | |
|---|---|
| Condenser tested | Adjustable oil Condenser No. II. |
| Capacity | 0·00266 microfarad. |
| Frequency | $1·11 \times 10^6$. |
| Nature of dielectric | Vaseline oil. |
| Surface of dielectric | 3,150 sq. cms |
| Thickness of dielectric | 0·238 cm. |
| Volume of dielectric | 750 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·10 | 0·052 | 0·500 | 0·00156 | 667 | 2·34 |
| 2·61 | 0·050 | 0·341 | 0·00150 | 455 | 1·97 |
| 2·40 | 0 060 | 0·346 | 0·00180 | 461 | 1·82 |
| 2·15 | 0·057 | 0·263 | 0·00171 | 351 | 1·63 |
| 1·80 | 0·060 | 0·194 | 0·00180 | 259 | 1·36 |

The observed power loss D in this condenser is related to the corresponding electric force E in the manner expressed by the formula $D=121E^{2·15}$, which fits in well with the above observations.

The observations in Tables IB. and IC. represent the results for the same condenser but adjusted for different capacities.

TABLE IB.

| | |
|---|---|
| Condenser tested | Adjustable oil Condenser No. II. |
| Capacity | 0·00140 microfarad. |
| Frequency | $1·55 \times 10^6$. |
| Nature of dielectric | Vaseline oil. |
| Surface of Dielectric | 1,575 sq. cms. |
| Thickness of dielectric | 0·238 cm. |
| Volume of dielectric | 375 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·20 | 0·119 | 1·218 | 0·00256 | 3,250 | 3·29 |
| 2·83 | 0·100 | 0·801 | 0·00215 | 2,140 | 2·91 |
| 2·58 | 0·104 | 0·692 | 0·00224 | 1,840 | 2·65 |
| 2·34 | 0·111 | 0·608 | 0·00239 | 1,620 | 2·40 |
| 1·99 | 0·123 | 0·487 | 0·00264 | 1,300 | 2·04 |

The above observations agree with the formula,

$$D=243·2E^{2·15}.$$

TABLE Ic.

Condenser tested ............... Adjustable oil Condenser No. II.
Capacity ............................ 0·00076 microfarad.
Frequency ......................... $2·14 \times 10^6$.
Nature of dielectric ............ Vaseline oil.
Surface of dielectric ............ 788 sq. cms.
Thickness of Dielectric......... 0·238 cm.
Volume of dielectric ............ 188 cub. cms.

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electro-static units. E. |
|---|---|---|---|---|---|
| 3·16 | 0·245 | 2·45 | 0·00382 | 13,020 | 4·34 |
| 2·87 | 0·247 | 2·04 | 0·00385 | 10,850 | 3·94 |
| 2·67 | 0·227 | 1·62 | 0·00354 | 8,630 | 3·66 |
| 2·45 | 0·247 | 1·48 | 0·00385 | 7,880 | 3·36 |
| 2·10 | 0·227 | 1·00 | 0·00354 | 5,330 | 2·88 |

The above observations agree with the formula,

$$D=550E^{2·15}.$$

The observations recorded in Tables ID., IE., and IF. refer to another vaseline oil condenser of the same type, taken for various capacities, and the exponential formula beneath each Table is that which best expresses the relation of loss to force in each case. The oils were not from the same sample, which probably accounts for the difference in the value of the exponents.

TABLE ID.

Condenser tested ............... Adjustable oil Condenser No. I.
Capacity ............................ 0·00271 microfarad.
Frequency ......................... $1·12 \times 10^6$.
Nature of dielectric ............ Vaseline oil.
Surface of dielectric ............ 3,150 sq. cms.
Thickness of dielectric......... 0·238 cm.
Volume of dielectric............ 750 cub. cms.

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electro-static units. E. |
|---|---|---|---|---|---|
| 3·14 | 0·072 | 0·710 | 0·00211 | 950 | 2·31 |
| 2·71 | 0·054 | 0·396 | 0·00161 | 530 | 1·99 |
| 2·45 | 0·036 | 0·216 | 0·00107 | 290 | 1·80 |
| 2·18 | 0·028 | 0·133 | 0·00083 | 180 | 1·60 |
| 1·81 | 0·028 | 0·092 | 0·00083 | 120 | 1·33 |

$$D=71E^{2·61}.$$

### TABLE IE.

| | |
|---|---|
| Condenser tested | Adjustable oil Condenser No. I. |
| Capacity | 0·00141 microfarad. |
| Frequency | 1·54 × $10^{6}$. |
| Nature of dielectric | Vaseline oil. |
| Surface of dielectric | 1,575 sq. cm. |
| Thickness of dielectric | 0·238 cm. |
| Volume of dielectric | 375 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·15 | 0·121 | 1·200 | 0·00262 | 3,200 | 3·24 |
| 2·78 | 0·122 | 0·943 | 0·00264 | 2,510 | 2·86 |
| 2·55 | 0·107 | 0·695 | 0·00232 | 1,850 | 2·62 |
| 2·30 | 0·096 | 0·508 | 0·00208 | 1,350 | 2·36 |
| 1·95 | 0·105 | 0·399 | 0·00227 | 1,060 | 2·00 |

$$D = 155E^{2\cdot61}.$$

### TABLE IF.

| | |
|---|---|
| Condenser tested | Adjustable oil Condenser No. I. |
| Capacity | 0·00076 microfarad. |
| Frequency | 2·11 × $10.^{6}$ |
| Nature of dielectric | Vaseline oil. |
| Surface of dielectric | 788 sq. cms. |
| Thickness of dielectric | 0·238 cm. |
| Volume of dielectric | 188 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·26 | 0·264 | 2·80 | 0·00417 | 14,900 | 4·59 |
| 2·96 | 0·250 | 2·19 | 0·00395 | 11,600 | 4·16 |
| 2·76 | 0·209 | 1·59 | 0·00330 | 8,470 | 3·88 |
| 2·53 | 0·210 | 1·34 | 0·00332 | 7,140 | 3·56 |
| 2·18 | 0·212 | 1·01 | 0·00335 | 5,380 | 3·06 |

$$D = 275E^{2\cdot61}.$$

These formulæ are all of the type $D = XE^{Y}$. It may be noticed that for the same oil and same condenser the exponent Y remains the same, but the multiplying factor X varies nearly inversely as the capacity or directly as the current density.

The observations recorded in Table II. for a glass plate condenser show exactly how the observations were taken in each case and treated so as to obtain the effective resistance $\rho$ of the condenser for a certain mean current A.

It will be seen that this equivalent condenser resistance is a function of the condenser current and increases with it. Hence that part of the total decrement due to the condenser losses increases with the condensers current. The bulk of the dielectric, its thickness, and the frequency were measured in each case as well as the inductance of the condenser circuit.

Table II.

*Example of Calculation.*

| | |
|---|---|
| Condenser tested ............ | Glass plate condenser in oil. |
| Capacity ........................ | 0·00130 microfarad. |
| Frequency ..................... | $1·68 \times 10^6$. |
| Inductance of circuit ...... | 6,920 cms. |

The circuit made up of No. 14 S.W.G. copper wire (diameter =0·203 cm.), using Dr. A. Russell's formula for the ratio of the high-frequency resistance to the steadycurrent resistance. In the case of the circuit used this ratio is 10·29, and hence since the steady current resistance of circuit=0·0350 ohm, the high-frequency resistance $R' = 10·29 \times 0·0350 = 0·360$ ohm.

The ammeter used was constructed of copper wire, and hence its resistance increased with the current passing (*see* Table), but this variation was determined and allowed for.

| High-frequency resistances in ohms. | | | | R.M.S. value of condenser current in amperes $A_1$. | $A_1^2$. | $A_1^2R$. | Difference of values of AR. | Difference of values of $A^2$. | Effective resistance of condenser in ohms. $\rho$ | Mean value of R.M.S. current in amperes A. |
|---|---|---|---|---|---|---|---|---|---|---|
| Inter- olated esist- ance $r$. | Ammeter resistance $r'$ | Circuit resistance $R'$. | Sum. R. | | | | | | | |
| 0 | 0·047 | 0·360 | 0·407 | 3·47 | 12·04 | 4·900 | 1·238 | 1·51 | 0·820 | 3·36 |
| 0·177 | 0·046 | 0·360 | 0·583 | 3·245 | 10·53 | 6·138 | 0·786 | 0·98 | 0·803 | 3·17 |
| 0·320 | 0·045 | 0·360 | 0·725 | 3·09 | 9·55 | 6·924 | 0·627 | 0·80 | 0·785 | 3·02 |
| 0·459 | 0·014 | 0·360 | 0·863 | 2·957 | 8·75 | 7·551 | 0·916 | 1·19 | 0·770 | 2·85 |
| 0·717 | 0·043 | 0·360 | 1·120 | 2·75 | 7·56 | 8·467 | 1·427 | 2·06 | 0·694 | 2·55 |
| 1·398 | 0·041 | 0·360 | 1·799 | 2·345 | 5·50 | 9·894 | | | | |

The observations recorded in Table III. were made in the above described manner, using a condenser made with dielectric of sheet ebonite and the whole immersed in insulating oil to prevent brush discharges. The results of the measurements show that the equivalent condenser resistance $\rho$ increases with the condenser current and that the condenser losses in microwatts per cubic centimetre D vary with the electrostatic force E in the dielectric, according to the law,

$$D=1{,}780E^{2\cdot87}.$$

Tables IV. and V. show measurements made with similar condensers of increasing capacity, but the factor X seems to decrease with the mean electric force.

TABLE III.

| | |
|---|---|
| Condenser tested | Ebonite in oil. |
| Capacity | 0·00084 microfarad. |
| Frequency | $2{\cdot}00 \times 10^{6}$. |
| Nature of dielectric | Ebonite. |
| Surface of Dielectric | 1,050 sq. cms. |
| Thickness of dielectric | 0·3 cm. |
| Volume of dielectric | 315 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 2·73 | 1·59 | 11·85 | 0·0265 | 37,700 | 2·87 |
| 2·63 | 1·52 | 10·52 | 0·0253 | 33,400 | 2·77 |
| 2·55 | 1·43 | 9·29 | 0·0238 | 29,600 | 2·69 |
| 2·45 | 1·39 | 8·34 | 0·0232 | 26,400 | 2·58 |
| 2·25 | 1·37 | 6·93 | 0·0229 | 22,000 | 2·37 |

The condenser losses in microwatts per cubic centimetre are represented by the formula,

$$D=1{,}780E^{2\cdot87}.$$

TABLE IV.

| | |
|---|---|
| Condenser tested | Ebonite in oil. |
| Capacity | 0·00170 microfarad. |
| Frequency | $1{\cdot}41 \times 10^{6}$. |
| Nature of dielectric | Ebonite. |
| Surface of dielectric | 2,100 sq. cms. |
| Thickness of dielectric | 0·3 cm. |
| Volume of dielectric | 630 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·24 | 0·850 | 8·92 | 0·0201 | 14,100 | 2·39 |
| 3·05 | 0·738 | 6·86 | 0·0174 | 10,900 | 2·25 |
| 2·90 | 0·715 | 6·01 | 0·0169 | 9,550 | 2·14 |
| 2·73 | 0·704 | 5·25 | 0·0166 | 8,350 | 2·02 |
| 2·41 | 0·657 | 3·82 | 0·0155 | 6,060 | 1·78 |

The results agree with the formula,

$$D=1{,}120E^{2\cdot87}.$$

TABLE V.

| | |
|---|---|
| Condenser tested | Ebonite in oil. |
| Capacity | 0·00258 microfarad. |
| Frequency | $1·14 \times 10^6$. |
| Nature of dielectric | Ebonite. |
| Surface of dielectric | 3,150 sq. cms. |
| Thickness of dielectric | 0·3 cm. |
| Volume of dielectric | 945 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·25 | 0·659 | 6·96 | 0·0193 | 7,380 | 1·96 |
| 3·02 | 0·577 | 5·26 | 0·0169 | 5,580 | 1·82 |
| 2·85 | 0·512 | 4·16 | 0·0150 | 4,400 | 1·72 |
| 2·65 | 0·504 | 3·54 | 0·0147 | 3,750 | 1·60 |
| 2·32 | 0·458 | 2·54 | 0·0134 | 2,690 | 1·40 |

The results fit in with the formula,

$$D = 1,000E^{2·87}.$$

Table VI. exhibits the results of measurements made with an air condenser, and show that though the energy losses are small they are by no means zero even in air condensers; but are immensely less than the losses in glass or ebonite dielectrics for equal volumes of dielectric but not much less for equal values of the electric force.

TABLE VI.

| | |
|---|---|
| Condenser tested | Air Condenser with zinc plates. |
| Capacity | 0·001093 microfarad. |
| Frequency | $1·63 \times 10^6$. |
| Nature of dielectric | Air. |
| Surface of dielectric | 58,800 sq. cms. |
| Thickness of dielectric | 5·0 cms. |
| Volume of dielectric | 294,000 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·22 | 0·049 | 0·508 | 0·00086 | 1·73 | 0·192 |
| 2·84 | 0·046 | 0·371 | 0·00081 | 1·26 | 0·169 |
| 2·60 | 0·039 | 0·264 | 0·00069 | 0·90 | 0·155 |
| 2·35 | 0·031 | 0·171 | 0·00055 | 0·58 | 0·140 |
| 1·99 | 0·026 | 0·103 | 0·00046 | 0·35 | 0·119 |

The results can be represented by the formula,

$$D = 590E^{3·5}.$$

Tables VII., VIII., and IX. give the results of measurements made with Leyden jars, such as are used in radio-telegraphy. The jars used in the experiments in Tables VII. and VIII. agree in giving a formula for the loss in terms of the electrostatic force very nearly the same, but the jar used in the experiments in Table IX. gave results which fit in best with a formula with larger value for X and lesser value for the exponent Y.

### TABLE VII.

| | |
|---|---|
| Condenser tested | Leyden jar. |
| Capacity | 0·00200 microfarad. |
| Frequency | $1 \cdot 30 \times 10^6$. |
| Nature of dielectric | Glass. |
| Surface of dielectric | 630 sq. cms. |
| Thickness of dielectric | 0·20 cm. |
| Volume of dielectric | 126 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·24 | 0·140 | 1·47 | 0·00359 | 11,700 | 3·31 |
| 2·83 | 0·118 | 0·945 | 0·00303 | 7,500 | 2·89 |
| 2·65 | 0·095 | 0·667 | 0·00244 | 5,300 | 2·71 |
| 2·34 | 0·074 | 0·405 | 0·00190 | 3,220 | 2·39 |
| 1·96 | 0·045 | 0·173 | 0·00115 | 1,370 | 2·00 |

The results agree with the formula $D = 77E^{4 \cdot 24}$.

### TABLE VIII.

| | |
|---|---|
| Condenser tested | Leyden jar. |
| Capacity | 0·00211 microfarad. |
| Frequency | $1 \cdot 27 \times 10^6$. |
| Nature of dielectric | Glass. |
| Surface of dielectric | 650 sq. cms. |
| Thickness of dielectric | 0·22 cm. |
| Volume of dielectric | 143 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·19 | 0·123 | 1·25 | 0·00323 | 8,750 | 2·88 |
| 2·79 | 0·084 | 0·653 | 0·00220 | 4,560 | 2·52 |
| 2·54 | 0·079 | 0·510 | 0·00207 | 3,570 | 2·29 |
| 2·29 | 0·053 | 0·278 | 0·00139 | 1,940 | 2·06 |
| 1·91 | 0·027 | 0·098 | 0·00071 | 690 | 1·72 |

The results can be represented by the formula,

$$D = 95E^{4 \cdot 24}.$$

TABLE IX.

| | |
|---|---|
| Condenser tested | Leyden jar. |
| Capacity | 0·00204 microfarad. |
| Frequency | $1{\cdot}29 \times 10^6$. |
| Nature of dielectric | Glass. |
| Surface of dielectric | 634 sq. cms. |
| Thickness of dielectric | 0·27 cm. |
| Volume of dielectric | 171 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·19 | 0·117 | 1·190 | 0·00302 | 6,970 | 2·39 |
| 2·80 | 0·103 | 0·808 | 0·00266 | 4,730 | 2·10 |
| 2·55 | 0·096 | 0·624 | 0·00248 | 3,650 | 1·91 |
| 2·30 | 0·084 | 0·444 | 0·00217 | 2,600 | 1·72 |
| 1·93 | 0·080 | 0·298 | 0·00207 | 1,740 | 1·45 |

The results fit in with the formula $D = 610E^{2\cdot75}$.

TABLE X.

| | |
|---|---|
| Condenser tested | Glass plate condenser in oil. |
| Capacity | 0·00130 microfarad. |
| Frequency | $1{\cdot}68 \times 10^6$. |
| Nature of dielectric | Glass. |
| Surface of dielectric | 309 sq. cms. |
| Thickness of dielectric | 0·17 cm. |
| Volume of dielectric | 52·5 cub. cms. |

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 3·36 | 0·820 | 9·25 | 0·0176 | 176,000 | 4·81 |
| 3·17 | 0·803 | 8·06 | 0·0173 | 154,000 | 4·54 |
| 3·02 | 0·785 | 7·15 | 0·0169 | 136,000 | 4·32 |
| 2·85 | 0·770 | 6·25 | 0·0166 | 119,000 | 4·08 |
| 2·55 | 0·694 | 4·50 | 0·0149 | 85,800 | 3·65 |
| 2·31 | 0·678 | 3·62 | 0·0146 | 69,000 | 3·30 |
| 2·16 | 0·676 | 3·16 | 0·0145 | 60,200 | 3·09 |
| 2·05 | 0·638 | 2·68 | 0·0137 | 51,100 | 2·94 |
| 1·93 | 0·664 | 2·47 | 0·0143 | 47,100 | 2·76 |
| 1·71 | 0·623 | 1·82 | 0·0134 | 34,700 | 2·45 |
| 1·17 | 0·550 | 0·754 | 0·0118 | 14,400 | 1·67 |
| 1·10 | 0·498 | 0·604 | 0·0107 | 11,500 | 1·57 |
| 1·04 | 0·458 | 0·495 | 0·0099 | 9,440 | 1·49 |
| 0·97 | 0·446 | 0·419 | 0·0096 | 7,990 | 1·39 |
| 0·85 | 0·370 | 0·266 | 0·0080 | 5,060 | 1·22 |
| 0·67 | 0·434 | 0·195 | 0·0093 | 3,720 | 0·96 |
| 0·62 | 0·406 | 0·154 | 0·0087 | 2,930 | 0·89 |
| 0·59 | 0·411 | 0·144 | 0·0088 | 2,740 | 0·84 |
| 0·55 | 0·430 | 0·129 | 0·0093 | 2,460 | 0·79 |
| 0·48 | 0·280 | 0·064 | 0·0060 | 1,220 | 0·69 |

The above observations agree with the formula $D = 3{,}720E^{2\cdot42}$ well down to $E = 1{\cdot}4$, then not so well.

The observations in Table X. concern the losses in a glass plate condenser and agree well with the formula $D=3{,}720E^{2·42}$, but those in Table XI. for another glass plate condenser with much thicker plates and those in Table XII. taken with a Moscicki condenser give results which will not fit in with any single formula of this type, but taken in stages for various ranges of E can be represented by several different formulæ of this form.

TABLE XI.

Condenser tested ............ Glass plate condenser in oil.
Capacity ........................ 0·0139 microfarad.
Frequency ...................... $0·51\times10^6$.
Nature of dielectric ......... Glass.
Surface of dielectric ......... 12,080 sq. cms.
Thickness of dielectric ...... 0·51 cm.
Volume of dielectric ......... 6,160 cub. cms.

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. |
|---|---|---|---|---|---|
| 9·90 | 0·357 | 35·0 | 0·0250 | 5,700 | 1·46 |
| 9·60 | 0·306 | 28·2 | 0·0215 | 4,580 | 1·41 |
| 9·30 | 0·286 | 24·7 | 0·0200 | 4,010 | 1·37 |
| 9·02 | 0·274 | 22·3 | 0·0192 | 3,620 | 1·33 |
| 8·39 | 0·256 | 18·0 | 0·0180 | 2,920 | 1·23 |
| 7·28 | 0·171 | 9·06 | 0·0120 | 1,470 | 1·07 |
| 6·97 | 0·141 | 6·85 | 0·0099 | 1,110 | 1·02 |
| 6·67 | 0·126 | 5·61 | 0·0088 | 911 | 0·98 |
| 6·41 | 0·117 | 4·81 | 0·0082 | 781 | 0·94 |
| 5·85 | 0·110 | 3·76 | 0·0077 | 610 | 0·86 |
| 3·14 | 0·079 | 0·779 | 0·0056 | 126 | 0·46 |
| 2·67 | 0·070 | 0·499 | 0·0049 | 81 | 0·39 |
| 2·39 | 0·062 | 0·354 | 0·0044 | 57 | 0·35 |
| 2·12 | 0·045 | 0·202 | 0·0032 | 33 | 0·31 |
| 1·75 | 0·016 | 0·049 | 0·0011 | 8 | 0·26 |

From $E=1·45$ to $0·9$. $D=1{,}200E^{4·41}$ points fit very well.
$E<0·9$. $D=903E^{2·61}$. points fit badly.

It is clear, therefore, that the energy losses in air and oil condensers made with certain oils are, relatively speaking, very small, but may yet amount to something of the order of several thousand microwatts per cubic centimetre for electrostatic forces of the order of 2·0 electrostatic units or 600 volts per centimetre. They are also determined by the current density. In the case of condensers made with glass or ebonite dielec-

TABLE XII.

Condenser tested .................. Moscicki condenser.
Capacity ................................ 0·004067 microfarad.
Frequency ............................ $0{\cdot}88\times10^6$.
Nature of dielectric .............. Glass.
Surface of dielectric ............... 930 sq. cms.
Thickness of dielectric ............ 0·15 cm. (very roughly).
Volume of dielectric ............... 139 cub. cms.

| R.M.S. value of condenser current in amperes. A. | Effective resistance of condenser in ohms. $\rho$ | Power dissipated in condenser in watts. W. | Decrement of condenser. $\delta$ | Power loss in condenser in microwatts per c.c. D. | R.M.S. value of electric force in dielectric in electrostatic units. E. | |
|---|---|---|---|---|---|---|
| 8·47 | 0·760 | 54·7 | 0·0270 | 394,000 | 8 38 | $D=1{,}000E^{2\cdot8}$. |
| 8·32 | 0·729 | 50·4 | 0·0259 | 362,000 | 8 23 | |
| 8·18 | 0·742 | 49·5 | 0·0264 | 356,000 | 8 09 | |
| 8·04 | 0·710 | 45·9 | 0·0252 | 330,000 | 7·95 | |
| 7·69 | 0·684 | 40·4 | 0·0242 | 291,000 | 7·60 | |
| 7·58 | 0·680 | 39·0 | 0·0242 | 281,000 | 7·50 | |
| 7·44 | 0·649 | 35·9 | 0·0230 | 258,000 | 7·35 | $D=7{\cdot}9E^{5\cdot2}$. |
| 7·30 | 0·614 | 32·7 | 0·0218 | 235,000 | 7·22 | |
| 7·17 | 0·585 | 30·1 | 0·0208 | 217,000 | 7·10 | |
| 6·78 | 0·371 | 17·0 | 0·0132 | 122,300 | 6·71 | |
| 6·30 | 0·386 | 15·30 | 0·0137 | 110,000 | 6·24 | |
| 6·13 | 0·344 | 12·91 | 0·0122 | 92,900 | 6·07 | |
| 5·97 | 0·327 | 11·64 | 0·0116 | 83,800 | 5·91 | |
| 5·82 | 0·304 | 10·29 | 0·0108 | 74,000 | 5·76 | |
| 5·44 | 0·234 | 6·92 | 0·0083 | 49,800 | 5·39 | |
| 3·15 | 0·262 | 2·600 | 0·0093 | 18,700 | 3·12 | |
| 2·82 | 0·186 | 1·479 | 0·0066 | 10,600 | 2·79 | $D=660E^{2\cdot8}$. |
| 2·59 | 0·163 | 1·093 | 0·0058 | 7,900 | 2·56 | |
| 2·35 | 0·149 | 0·823 | 0·0053 | 5,900 | 2·32 | |
| 1·99 | 0·138 | 0·546 | 0·0049 | 3,900 | 1·97 | |

trics these internal energy losses may become very large for electrostatic forces attaining a value not even larger than 10, and hence contributing greatly to raise the decrement in any oscillatory circuits of which these condensers form part. Different kinds of glass seem to vary very much in this respect, and this points to the great importance of testing Leyden jars and glass plate condensers intended to be used as radiotelegraphic transmitters, especially for dielectric loss as well as for capacity. With the arrangements above described this can be most easily and accurately carried out, and that glass may be selected which is best for the purpose. The experiments also show that for experimental work condensers made with oil as dielectric are best, whilst for large scale condensers air as a dielectric gives a small energy loss, provided the electrostatic force is not raised to a value at which brush discharges begin to produce sensible ionisation of the air. If this is the case the internal losses may become considerable, even for air condensers. The moral to be drawn from the above described observations is that in the case of condensers used in Radiotelegraphy the current density in the condenser should be kept as small as possible to reduce the dielectric losses. As regards the sources of these energy losses in solid or liquid dielectrics some part of the loss is probably due to actual conduction. Some part may be due to brush discharges and the remainder to dielectric losses, which are generally referred to as dielectric hysteresis, though this term has never been accurately defined. In a Paper on "The Conversion of Electric Energy in Dielectrics," *see* Physical Review, Vol. IV., May, June; Vol. V., July, August, 1897, Prof. R. Threlfall discusses very fully the sources of energy loss in dielectrics, and found that when ellipsoids of dielectrics were suspended in a revolving electric field and the dielectric hysteresis thus measured the energy absorption could be expressed in terms of the internal electric force by an exponential formula such as that we have used. The formula, however, in itself conveys no physical meaning, but is merely a convenient method of recording the observations.

## ABSTRACT.

In this Paper an arrangement of apparatus is described for the purpose of measuring the internal energy losses in condensers traversed by high frequency (H.F.) currents. It is shown that these energy losses in condensers may be considered as if they were due to a resistance loss in a hypothetical resistance in series with the condenser, the condenser itself being supposed to have a perfect non-dissipative dielectric of the same dielectric constant. This hypothetical resistance is not constant, but is a function of the condenser current. The experiments were conducted by the use of a special form of impact dischargers comprising two flat plates immersed in oil, one stationary and the other revolving at a high speed. This discharger was placed in series with a primary circuit and condenser, and H.F. oscillations were set up in the primary having any desired frequency. A secondary circuit loosely coupled consisted of a wire whose H.F. resistance could be determined, the condenser to be examined and a hot-wire ammeter and variable resistances. The measurements consisted in observing the reading of the ammeter, and then changing the current created in the secondary circuit by a small amount by adding a known resistance which altered the decrement of the circuit, but not its inductance. From these readings an equation is obtained for the hypothetical condenser resistance. It is shown that the product of

the square of the secondary current $A^2$ and the total resistance R of the secondary circuit is constant, and hence that the unknown condenser resistance $\rho$ can be found from observations of the change in $A^2$ when an additional resistance $r_1$ is interpolated in the condenser circuit. The energy loss in the condenser is then $A^2\rho$ watts. Condensers with various dielectrics, air, oil, glass and ebonite, were examined, and the dielectric energy losses D are stated in micro watts per cubic centimetre of dielectric for given values of the electric force E. It is shown thus D can be expressed as a function of E in the form $D = XE^Y$, where X is a constant depending on the current density, and Y is a constant depending on the nature of the dielectric. For oil and air these power losses are relatively small, but for glass and ebonite large. The necessity for measuring these losses in the case of radiotelegraphic condensers is emphasised.

[The discussion on this paper was combined with that on the following paper on "Resonance Curves," by the same Authors.]

## XII. *Some Resonance Curves taken with Impact and Spark Ball-Dischargers.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S.* *and* G. B. Dyke, *B.Sc., A.M.Inst.E.E.*

Read January 27, 1910.

In the course of the experiments on the energy losses in condensers traversed by high-frequency currents described in the previous Paper,* we had occasion to take many measurements with the cymometer of the frequency and inductance of the secondary circuit used in which oscillations were created by a primary circuit used both with the impact and with the ordinary spark-ball discharger. It then became an easy matter to take sufficient observations to delineate a complete resonance curve in each case. As these curves give a very interesting insight into the operations taking place in the secondary circuit and as they confirm the ordinary theory of the oscillation transformer, and, moreover, exhibit the striking difference between the action of an impact and of an ordinary spark-ball discharger, we have delineated for each such case a family of resonance curves taken over a certain range of frequency and with various degrees of coupling. The arrangements for experiments were just as described in the previous Paper, only the secondary circuit was moved to or from the primary to vary the coupling and the cymometer circuit was in all cases loosely coupled to the secondary circuit. The resistance of the secondary circuit was varied so as to give in one case a small decrement of about 0·01 per semi-period and in the other case a much larger decrement of 0·09 per semi-period. The spark-gap was 1 mm. in length and acted upon by a steady blast of air to produce uniform discharges.

The current in the cymometer circuit was measured by a hot-wire thermoelectric ammeter. In the diagrams given the vertical ordinates represent this cymometer current in milliamperes. The decrement of the cymometer itself was about 0·03. The impact discharger used was that described in the previous Paper on "Energy Losses in Condensers traversed by High-Frequency Currents" (*loc. cit.*).

The curves in Fig. 1 delineate the resonance curves taken with the ordinary spark-gap and with such resistance in the secondary circuit as to give a semi-period decrement of 0·01. The different curves correspond to different degrees of coupling $k$ of the primary and secondary circuits, as shown in the diagram. When the coupling is about 33 per cent. the resonance curve 1 shows three well-marked maxima or humps. The frequency corresponding to the middle hump is the natural free period frequency of the secondary circuit. The frequencies of the other two maxima on either side correspond to the two oscillations which are created by the reaction between the primary and secondary circuits.

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* "The Measurement of Energy Losses in Condensers traversed by High-Frequency Electric Oscillations." By J. A. Fleming and G. B. Dyke.

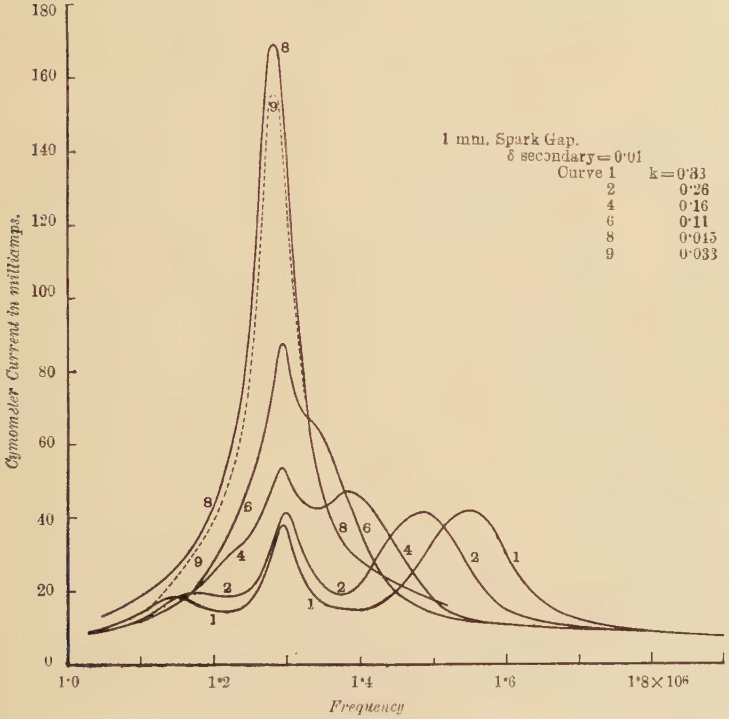

FIG. 1.

These three frequencies have such values that the square of the free frequency is the mean of the squares of the other two frequencies. This confirms and agrees with theory; also the calculated value of $k$ agrees fairly well with that deduced from the two frequencies. As the coupling is reduced respectively to 26 per cent. and 16 per cent., we have the resonance curves 2 and

4 from which it is seen that the amplitudes of the two period oscillations are decreasing and the free oscillation survives. When the coupling is reduced to 11 per cent. there remains only a very slight indication of a forced oscillation. If the coupling is weakened still more we have only the free oscillation left and the secondary current comes to a maximum for a coupling of about 5 per cent. Beyond that weakening the coupling decreases the maximum current in the secondary circuit. If we take off the values of the maximum cymometer currents in each curve and plot these as ordinates of another curve to abscissæ representing the coupling, as done in Fig. 5, we have a curve (the upper dotted curve marked "spark") which shows us how the current in the secondary circuit varies with the coupling between the secondary and the primary. It has a well-marked maximum value for about 5 per cent. coupling.

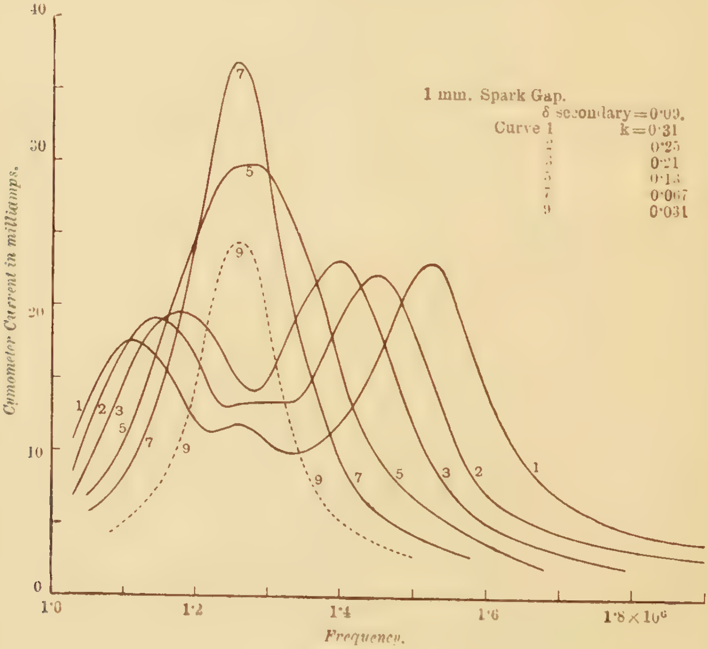

FIG. 2.

The family of resonance curves in Fig. 2 is taken with the same spark-ball discharger in the primary circuit but with resistance added to the secondary circuit to increase its semi-period decrement to 0·09. It will be seen that for a coupling of 31 per cent. there is a trace of a free vibration, but the predominant oscillations in the secondary are the two-period oscillations. As the coupling is weakened the free oscillation makes its appearance, but it is not until the coupling is weakened to 13 per cent. that the free oscillation alone survives.

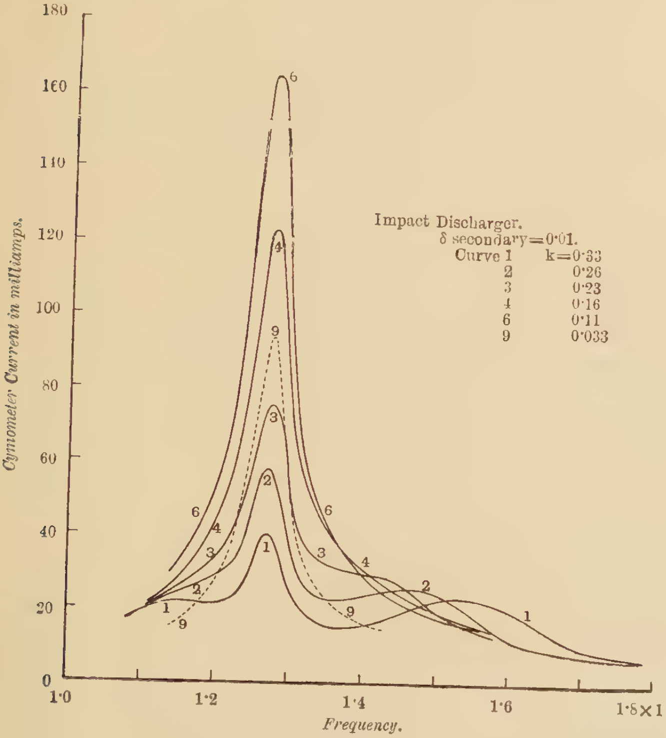

Fig. 3.

When the coupling is reduced to about 7 per cent. we have the maximum currents in the secondary circuit.

These observations were then repeated, using the impact discharger, and the same general phenomena are observed, as shown by the curves in Figs. 3 and 4, with the difference that for a secondary decrement of 0·01 the free secondary oscillation makes its appearance when the coupling is as strong as 25 per cent., and when the secondary decrement is 0·09 the free secondary oscillation alone appears even when the coupling is 20 per cent.

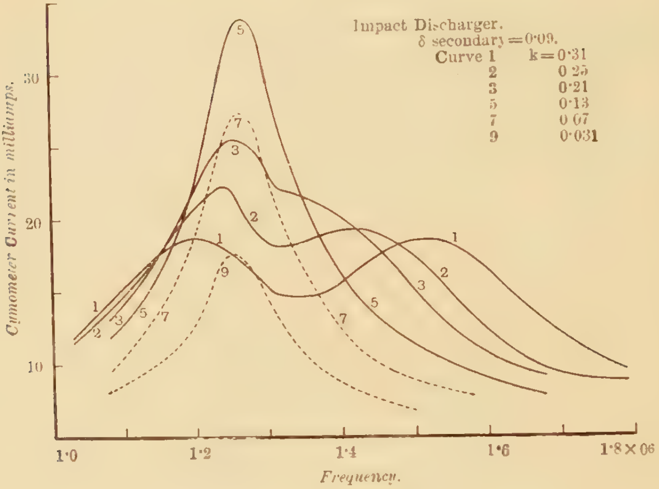
Impact Discharger.
δ secondary = 0·09.
Curve 1 k = 0·31
2 0·25
3 0·21
5 0·13
7 0·07
9 0·031
Cymometer Current in milliamps.
30
20
10
1·0
1·2
1·4
1·6
1·8×06
Frequency.
1
2
3
5
7
9

FIG. 4.

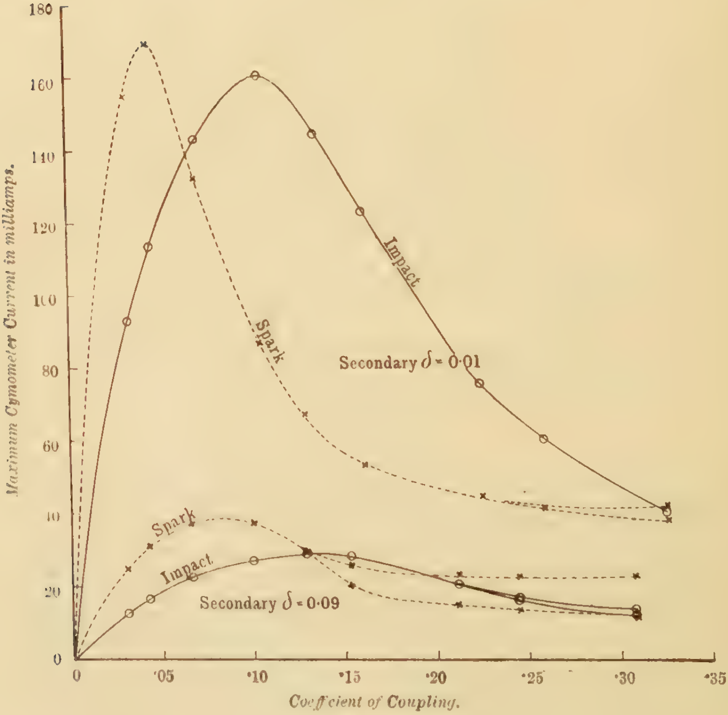
Maximum Cymometer Current in milliamps.
180
160
140
120
100
80
60
40
20
0
0
·05
·10
·15
·20
·25
·30
·35
Coefficient of Coupling.
Impact
Spark
Secondary δ = 0·01
Spark
Impact
Secondary δ = 0·09

FIG. 5.

The firm line curves in Fig. 5 represent the variation of the secondary current as the coupling between primary and secondary varies. The advantages of the impact dischargers may therefore be summed up by saying that with it we can have closer coupling between the primary and secondary circuit and yet retain the advantage of having no other oscillations set up in the secondary circuit than its natural free oscillation. This, however, omits all consideration of the number of the discharges and of the efficiency of the transmitter as a whole. Closeness of coupling always is an advantage in increasing plant efficiency, provided we can secure it without setting up forced oscillations in the secondary which annihilate the sharp peak of the resonance curve and therefore render sharp tuning between a receiving station and that particular transmitter impossible.

## ABSTRACT.

In the course of the experiments described in the previous Paper on the measurement of energy losses in condensers a large number of measurements had to be made with the cymometer of the frequency of oscillations in, and the inductance of, the secondary or condenser circuit. It was then an easy matter to draw complete resonance curves in each case, and this has accordingly been done with both the impac and spark ball dischargers in the primary circuit and for various resistances in the secondary circuit. The results are interesting as showing exactly what takes place in each case in the primary circuit. If we are using the spark ball discharger, and if the primary and secondary circuits are coupled with various degrees of coupling, then, for any close coupling. we find on the resonace curve three peaks which correspond, respectively, as regards frequency, with the free oscillation period of the secondary and with the two-period oscillations set up by the reaction of the secondary upon the primary circuit. As the coupling is weakened the double-period oscillations die out, and only the free oscillation of the secondary survives. There is always a certain coupling, not far from 10 per cent., which gives the maximum current in the secondary circuit in the form of a free oscillation. If the secondary circuit is more highly damped, then the two-period oscillations are more strongly marked, and the maximum free period oscillation has a lesser maximum value.

If we are using an impact discharger the double-period oscillations are only apparent when the coupling exceeds about 30 per cent., and die away with a very little reduction in the coupling, leaving the predominant free secondary oscillation as the survivor. These curves show how very quickly the primary spark is quenched when using the impact discharger. If the maximum secondary current is set up as ordinates in terms of the coupling as abscissa we obtain curves which rise up quickly to a maximum value and fall again, and which indicate that the maximum value of the secondary current is determined both by the coupling and the secondary decrements.

## DISCUSSION.

Mr. A. **Campbell** remarked that the want of constancy of the effective resistance as the current is changed seemed to be out of agreement with the results of other recent observers. Various condensers have been tested at the National Physical Laboratory at frequencies of 800 and 2,000 ∿ per second by Carey Foster's method. No divergence from the square law is found from 0·5 to 6 volts. It would be surprising if the square law did not hold at 500,000 to 1,000,000 ∿ per second. The formula employed by Dr. Fleming to give $\rho$ can scarcely be expected to give accurate results, since so much depends upon the small difference $A_1 - A_2$. Monasch ("Electrician," June and July, 1907) has shown most conclusively that for frequencies about 80 ∿ per second the losses in glass and ebonite condensers (and also in cables) were exactly proportional to the square of the effective voltage between 1,000 and 8,000 volts. He has pointed out that the heating of the dielectric by the power spent in it very often appears to cause variation from the square law. Stöckly ("La Lumière Electrique," July, 1909) has recently found that the decrement of a Moscicki condenser rose from 0·0165 at 10°C. to 0·021 at 40°C.

Mr. W. **Duddell** expressed surprise that with low potential gradients the energy loss was not proportional to $V^2$. Increase of the losses in a rather peculiar way with the frequency—not necessarily proportional—was to be expected. He cited his experiments made with high frequencies on an aerial at Bushy. The easiest method of finding the losses was to put in a resistance to bring the current down to half value. This method was justified with an air condenser.

He remarked that he thought it was the *voltage* that was induced in the secondary that was fixed, and not the amount of *energy* that was transferred.

He pointed out that we were never certain as to how much energy was lost in each part of the secondary circuit, and drew attention to the importance of knowing just where all the energy losses took place, as then we could soon begin to make our apparatus efficient.

Mr. E. H. **Rayner** suggested that a calculation of the power factors of the condensers would have given results which afforded considerably more information at a glance. If the losses varied as the square of the voltage, since they are small, the power factor would be constant at all voltages; if the losses increased faster than the square of the voltage the power factor would rise; if at a slower rate it would fall. The experimental results were not accurate enough to determine the constants of the formula employed by the authors. He had analysed the results given in the Paper, and found that some cases showed a diminishing power factor, giving an index less than 2, while the authors had given 2·15. He drew attention to some results which he had obtained on a gutta-percha cable showing a power factor practically constant from 10 volts to 4,000, demonstrating an exceedingly close approximation to the square law over a very great range of electric force.

Mr. F. **Jacob** remarked he had been for some time at Messrs. Siemens Bros. & Co.'s Cable Works, testing the leakance of gutta-percha cores and other dielectrics, or in other words the effective resistance of condensers, not certainly at the extra high periodicities of Prof. Fleming's most interesting and important results, but at such frequencies as obtained in telephonic transmission, say from 750 ∿ to 2,000 ∿. Their mode of testing was the bridge method, and the results obtained were correct within about 1 per cent. Two adjacent arms of the bridge are occupied by two equal resistances, the other two arms containing respectively the condenser K whose leakage resistance S is to be tested, and a variable condenser C, whose leakage is zero, in series with an adjustable resistance R. An alternating current and a telephone were employed. When balance is established

$$\mathrm{R}-\frac{1}{ip\mathrm{C}}=\frac{1}{\mathrm{S}-ip\mathrm{K}}.$$

Simplifying, and neglecting $S^2$ in comparison with $p^2K^2$ we find $K=C$ and $S/K=p^2RC$.

Hundreds of determinations on different materials show that for a given material and frequency S/K is a constant, and within ordinary electrical pressures independent of the same, and in some instances in direct proportion to the periodicity, but in others a little higher. His pressures corresponded very nearly to those given by Prof. Fleming's experiments.

S/K for air by his experiments was 0, and also for a mica condenser if of special construction.

In connection with Mr. Rayner's remarks, he stated that at 750 ∼ the value of S/K for the gutta-percha core of the loaded channel telephone cable was 114, and at 2.000 ∼, 0·320.

He hoped Prof. Fleming would apply his thermometric differential detector in a similar way, and thus obtain more directly the effective resistances of the condensers he had recently determined by comparison with an air condenser for which his experiments showed the effective resistance to be very small.

He had, on the assumption that the value S/K is proportional to the frequency, reduced Prof. Fleming's results to Messrs. Siemens standard form of S/K. He found no variation with electrical pressure. Below are excerpted values at the different frequencies comparing pairs which are at the same pressure.

EBONITE.

| E. | C. | R. | $10^6$ ∼ | S/K. | C 2,000 ∼ |
|---|---|---|---|---|---|
| 2·37 | 0·00084 | 1·37 | 2·0 | 182,500 | 182 |
| 2·39 | 0·00170 | 0·850 | 1·41 | 114,600 | 163 |
| 2·02 | 0·00170 | 0·704 | 1·41 | 94,900 | 135 |
| 1·96 | 0·00258 | 0·659 | 1·14 | 87,700 | 154 |
| 1·78 | 0·00170 | 0·657 | 1·41 | 88,600 | 126 |
| 1·82 | 0·00258 | 0·577 | 1·14 | 76,800 | 135 |

The last column of values reduced to 2,000 ∼ shows three values nearly identical and three differing, but no doubt further experiments will elucidate the cause.

At 2,000 ∼ for the best ebonite we get a value of 71, and at 800 of 29, which is practically proportional to those periodicities, but there are very different qualities of ebonite, and as there is oil also to take into account, and he did not know what part of the capacity is due to the two dielectrics in question.

The resistances found by Prof. Fleming are anyhow the effective values of these condensers, though we may not be able to separate the portion that is due to the dielectric alone. He found that the ordinary values of dielectric resistance measured after a minute, give no indication as to value of S/K.

Mr. G. L. ADDENBROOKE said his knowledge of high-frequency currents was not enough to enable him to speak authoritatively as to the accuracy of Dr. Fleming's method, but he had found that the form of the wave was of considerable importance, especially where the ammeter reading was relied on at all. Nevertheless he could support Dr. Fleming and Mr. Dyke in one point. He had himself found losses in air condensers after they had been in use for a time, and the loss increased until they might have a power factor of 1 per cent., that is in cases where the plates were about ⅛ in. apart. He had intended calling attention to this before seeing Dr. Fleming's Paper.

Next as to the experiment in Dr. Fleming's formula where heating was not sensible, he had always found the losses increased very nearly as the square of the current, but in a Paper sent into the Institution of Electrical Engineers some time since, the publication of which he had kept back, he had shown that excessive and long continued pressure might cause heating and greatly increase the losses, and more recently he had found the converse to be the case, that the losses were considerably increased by heating the dielectric, though in different degrees in different dielectrics.

Dr. W. H. Eccles, referring to the Paper giving resonance curves got from a secondary circuit excited by a primary containing either an ordinary spark or a quenched spark, said there was an apparent conflict between the usual theory and the facts disclosed by these curves. A pair of coupled circuits of equal frequency form in general a system of two degrees of freedom, which therefore possesses two periods of oscillation lying one on each side of the natural period of either circuit. The fact that Dr. Fleming's curves show, for some degrees of coupling, three humps instead of two, indicates that the pair of circuits does not remain a double system throughout the oscillation. One circuit disappears at a more or less early stage of the process, that is to say, the primary spark dies out and virtually removes the primary circuit from the combination, so that thereafter the secondary circuit vibrates alone. Usually it is assumed in the case of ordinary sparks between two balls that the spark endures almost to the end of the train of oscillations, and in the case of quenched sparks it is assumed that the primary spark is stopped at the end of one or two half swings. Dr. Fleming's curves show that both of these assumptions would be wrong in the case of his apparatus: the ordinary spark dies out sooner and the quenched spark later than is usually thought. The former fact is perhaps accounted for by the spark being placed in a blast of air. The curves show that in both cases the closeness of coupling has a profound effect on the length of life of the spark; with loose coupling the spark dies out much earlier than with close coupling.

Prof. G. W. O. Howe said he would like to ask Dr. Fleming what he meant exactly by "a dead-beat spark." Dr. Fleming used this expression several times in speaking of the impact discharger. It was stated in the Paper that "it is a dead-beat discharge," and "the spark is damped out of existence instantly," and again, "the discharge is practically non-oscillatory." He did not believe that this was an accurate discription of what actually took place in the primary circuit of a so-called impact discharger.

The primary and secondary circuits having been tuned to exact resonance and coupled together, the energy set oscillating in the former circuit by the breaking down of the spark-gap is gradually transferred to the secondary. The rate of this transfer depends on the tightness of the coupling, but with the tightest coupling used by Prof. Fleming, viz. 0·33, it would take a cycle and a half to transfer all the energy to the secondary circuit. If the spark was quenched under these conditions one might perhaps be justified in speaking of the transfer of energy as an impact. The resonance curves in Fig. 3 show that the spark is not quenched at this point, but that the energy surges backwards and forwards several times just as it does in an ordinary spark-gap in air. On loosening the coupling the transfer of energy takes place less rapidly, and the interval during which the primary current has a small value may be so long that the spark is quenched sooner. To enable the spark to be quenched at the first opportunity, that is, at the moment when the energy has been transferred to the secondary circuit for the first time, it appears from Fig. 1 to be necessary to reduce the coupling to 0·11. With this weak coupling, however, about 5 complete cycles are required to transfer the energy from the primary to the secondary circuit. If the oil-gap be replaced by the 1 mm. air-blast gap, Fig. 1 shows that the coupling has to be still further reduced to 0·045 before the spark is quenched at the first opportunity. The air-gap spark is more persistent than the oil-gap spark, but the difference between the two is merely one of degree.

The train of waves that follows each spark may be divided into two parts; during the first part we have beats between two coupled circuits, giving the two outside humps on the wave-meter curves; during the second part we have a single oscillatory circuit, giving the middle hump in the curves.

In view of the nature of the phenomena involved, he did not think we could speak of a dead-beat spark or discharge, but, under the best conditions as regards coupling, we may have a dead-beat surge of energy from the primary to the secondary circuit. For this reason he thought that the term "quenched spark" was preferable to "impact" or "shock excitation."

Mr. W. C. S. PHILLIPS remarked that he had found Prof. Fleming's Paper on "The Energy Losses in Condensers" of great interest, because it seemed to have some bearing on the results of some comparative tests of the capacities of condensers which he made some time ago.

Each condenser was in turn made part of an oscillating circuit by putting it in series with a known constant inductance, and the frequency of the oscillations was measured by means of a Fleming cymometer.

From a knowledge of the inductance of the oscillating circuit and its oscillation constant, the capacity of each condenser was obtained. When using a parallel plate condenser immersed in oil, a neon tube was quite satisfactory for the determination of the oscillation constant. Without in any way altering the adjustment of the two circuits the parallel plate condenser was replaced by a leyden-jar of equal capacity. The cymometer current was now so weak that it was impossible to detect the position of resonance by means of the neon tube. As care was taken not to alter the coupling of the circuits, nor to introduce any resistance by bad contact into the oscillating circuit, he could only explain the discrepancy by supposing that there was a considerable loss of energy in the condenser. The current supplied to the induction coil used for the production of the spark discharge was, of course, kept constant.

Dr. FLEMING, in reply, said he noticed that surprise had been expressed by several speakers that the dielectric losses are not more nearly proportional to the square of the voltage or electric force. With one or two exceptions, their experiments show a loss varying as some power between 2 and 3. He did not think that we could infer from experiments at low frequency what was likely to take place at high frequency, because there are some special sources of loss in the latter case. Thus, for instance, losses due to brush discharges from points and edges of the metal coatings are by no means negligible. It was found necessary to remove a brass indicating pointer from the variable oil condenser and replace it by an ebonite one, because a very obvious electric brush formed on the metal tip. These ionization losses might vary as some power greater than 2.

It is certain that these brush discharge losses are always present in the case of high-frequency measurements. The slight departure from the square law in the case of many of the results such as 2·15 instead of 2·0, may well be due to unavoidable errors of experiment. The method employed required care in the calibration and use of the high-frequency ammeter used. The objection raised by Mr. Campbell to the fact that we have to take the difference of the squares of two nearly equal currents is not so serious as it seems. The thermoelectric ammeter used is associated with a Paul single pivot galvanometer, and the scale of this last instrument is equidivisional, and the readings proportional to the mean-square value of the alternating current passing through the wires. Hence the square value was read at once. He was glad to find Mr. Jacob's interesting results so nearly in accord with his own, for ebonite condensers.

The observations made by Mr. Phillips on the difference between the effects of a leyden-jar and an oil condenser of the same capacity, when used with a cymometer and neon tube, were quite in accord with his own experience. They are undoubtedly a proof that there is a larger dissipation of energy in the case of the leyden-jar.

Mr. Rayner's careful analysis of our Paper does much to show that there is a high degree of probability that the principal part of the dielectric loss is, at any rate, proportional to the square of the electric force. When dealing with high-frequency oscillations there is, of course, much more difficulty in getting very accurate results than with low frequency. The observations we have taken would be impossible altogether without the use of an impact discharger, but it may be that the oscillations in the secondary circuit are not absolutely the free oscillations, and it is upon the assumption that they are free that our theory of the apparatus is based. As regards the last Paper on "Resonance Curves," Dr. Eccles puts the facts very concisely. There is undoubtedly a back effect produced, by the secondary oscillations, on the primary circuit, which results in a prolongation of the primary spark when close coupling is used, and this results in the appearance of the two-period oscillations superimposed on the free period oscillation in the secondary circuit.

Replying to Prof. Howe as to the sense in which the phrase "dead-beat spark" is used, Dr. Fleming said that it was intended to imply a spark which is very quickly quenched. The resonance curves given in their Paper prove that the quenching of the spark is not absolutely instantaneous, but that even with the so-called "impact discharger" the primary spark does endure for a very short time. If, however, the coupling does exceed a certain value, the result is to set up in the secondary only the free oscillations, and the advantages of the impact discharger are that this condition pertains for a closer coupling than is the case with the ordinary spark ball discharger. He agreed that the term "impact discharger" is perhaps not the best that could be used, but it is well understood, and as they have used it, it communicates to the secondary circuit just one non-reverberatory wave of energy which is then dissipated in the secondary circuit as heat.

## XIII. *On Flames of Low Temperature Supported by Ozone.*

*By the Hon.* R. J. Strutt, F.R.S., *Professor of Physics, Imperial College of Science, S. Kensington.*

Received Jan 16, 1911.

In a Paper read before the Physical Society on Nov. 25, 1910, it was shown that the yellow afterglow often observed when an electric discharge has passed through a vacuum tube containing air, is due to the mutual reaction of nitric oxide and ozone, each formed in the discharge. The glow can, in fact, be imitated by allowing nitric oxide, chemically prepared, to mix with a stream of oxygen which has been through a vacuum tube at low pressure.

The intention was expressed of examining the action of ozone on other oxidisable gases, to see whether any analogous flame would be produced. The results of such experiments will now be described.

The apparatus employed has been modified to suit the conditions of particular cases, but the form shown in the figure may be regarded as typical. A vacuum discharge tube A is fed with a stream of oxygen through the stopcock B. The ozone thus produced issues through the jet C into the wider tube D, where it meets with a current of any gas under investigation, introduced through the stopcock E. The powerful air pump which is used to maintain the gas currents is attached to the side tube G. F is a quartz window, cemented on with sealing wax, through which the spectrum of the flame produced in the space D can be photographed, without losing the ultra-violet portion of the spectrum.

When the vapours of volatile liquids were to be investigated the apparatus was the same, with a bulb containing the liquid attached to E instead of the supply of gas. It is not advisable to allow such vapours to find their way into the pump, where they may dissolve in the oil used to lubricate it, and prevent the attainment of a good vacuum. To protect the pump, the gases were drawn through a tube loosely packed with copper oxide, on their way to it. This tube was heated to redness, and effectually oxidised any of the vapour which might be in excess of the ozone.

When it was desired to work with the vapour of a volatile solid, fragments were simply placed in the tube D, the window F being temporarily removed to admit them.

In photographing the spectra of the flames a little stray light from A was liable to find its way by reflection into D, and some of the brighter lines of the ozonising discharge were recorded, but these were readily eliminated by comparison with a direct photograph of the spectrum of this discharge. A small single prism quartz spectrograph by Hilger was employed, and good photographs of the spectra of the various flames were usually obtainable with an hour's exposure or less.

To observe the temperature of the various flames, a similar apparatus was employed, but with the tube D sufficiently prolonged to contain a mercury thermometer, which was placed so that its bulb was enveloped in the flame.

The phenomena obtained in individual cases will now be described.

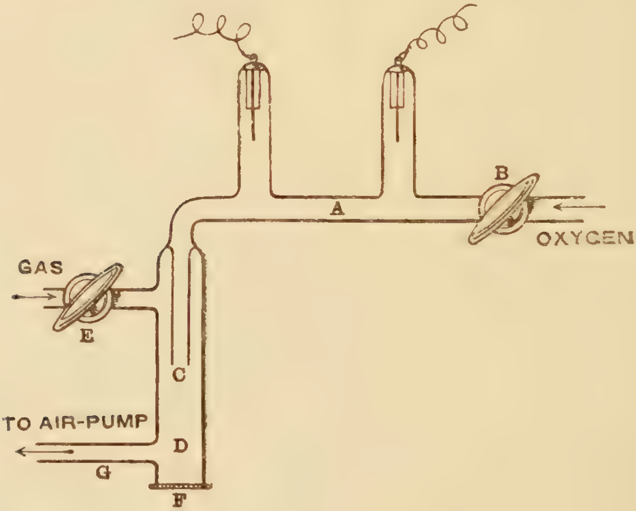

FIG. I.

*Sulphuretted Hydrogen.*—A stream of this gas mixed with ozone gave a sky-blue flame free from the " muggy " appearance of the nitric oxide flame. The temperature recorded was 70°C. This flame, in common with all those to be described, was much inferior in luminosity and volume to that given by nitric oxide. The chemical action takes place in a restricted region, and a sensible rise of temperature is obtained. 70°C. was reached, but this of course depends on the exact conditions of pressure.

The spectrum ranged from λ2,300 to λ4,800. Over most of this range it was continuous, but in the extreme ultra-violet, from λ2,300 to about λ2,600 bands could be distinguished, and it is probable that they extend further, though masked by the continuous spectrum.

*Sulphur.*—Instead of feeding a stream of sulphuretted hydrogen into the stream of ozone, a fragment of solid sulphur was placed in its path, at D (Fig. 1). A few seconds after the ozone current was started a blue flame, indistinguishable from the former one, was seen to play round the sulphur. As the experiment continued, the sulphur became warm, and melted, showing visible signs of ebullition, of course, at the low gaseous pressures at which the experiment was conducted.

The slight interval of "hang fire" before the glow commences no doubt represents the time required for the sulphur to get hot enough to give an appreciable amount of vapour. If the electric current was turned off, the glow at once ceased in this case as in all the others. The combustion only occurs at these low temperatures and pressures in ozone. It is not maintained when oxygen is substituted.

The spectrum of the sulphur glow was photographed, and was indistinguishable from that of the sulphuretted hydrogen glow. Mr. Eagle kindly photographed the spectrum of sulphur burning in air. It is roughly co-extensive with the former spectrum, but does not show the same system of bands. With atmospheric combustion, bands occur at the visible end of the spectrum.

It appears that sulphur burned in either way gives the same continuous spectrum ; superposed on this is a system of bands different in the two cases. Further study of these bands must be left to specialists in spectroscopy.

*Carbon Disulphide* gave a blue flame, apparently the same as sulphur or sulphuretted hydrogen. The spectrum was not examined.

There can, I think, be no doubt that the blue flame observed in these experiments is the same as the blue after-glow observed by Newall * with the electrodeless discharge, which he connected with oxygen and sulphur. The connection is, in fact, the simplest possible, the glow being produced by combustion of sulphur in ozone.

*Sulphur Dioxide* gave no glow with ozone.

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* "Proc." Camb. Phil. Soc., Vol. IX., p. 295, 1897.

*Selenium* in the ozone stream gave no glow in the cold; but when gently heated it was surrounded by a not very conspicuous glow. This was not examined further.

*Arsenic* gave no glow and was not visibly acted on.

*Iodine* gives an orange glow, redder than that due to nitric oxide. It is oxidised to a yellowish white solid, presumably iodine pentoxide.

The glow is fairly bright to the eye, but no spectrum was recorded on the plate by a 20 minutes' exposure. Visual observations showed that the spectrum was confined to the red and yellow regions, which accounts for the failure to photograph it. So far as could be observed, the spectrum was continuous.

*Hydrogen, Ammonia, Carbon Monoxide* gave no glow. *Nitrogen* and *nitrous oxide* were shown in the former Paper (*loc. cit.*) to give none.

*Acetylene*, in contrast to most hydrocarbons gives a fairly conspicuous glow, of bluish green colour. A thermometer immersed in this glow indicated 100°C. under the particular conditions of the experiment. The spectrum consisted of the Swan bands, together with the hydrocarbon band at $\lambda$4,315, and is thus identical with that of the inner cone of a Bunsen flame. The colour of the flame can be regulated by the acetylene supply. With a small supply it is green, the Swan bands being predominant. With an increased supply the flame contracts, the ozone being now consumed before it has got far from the jet where it issues into the acetylene. At the same time the colour becomes bluer, and the hydrocarbon band brighter, relatively to the Swan bands, than in a Bunsen flame. The ultra-violet bands of water vapour are not obtained from the acetylene glow.

The chemical action occurring in this glow has as yet only been partially studied. The gases from the glow were drawn through a U-tube cooled in liquid air on their way to the pump, with a view to retaining condensible products. When the glow had been maintained for some time this U-tube was detached, and the gases condensed in it collected through a Toepler pump, for analysis. The following experiment was typical:—

| | |
|---|---|
| Acetylene admitted into the apparatus | 17·3 c.c. |
| Total gas collected in U-tube | 11·0 c.c. |
| After absorption with caustic potash | 8·0 c.c. |

The remainder of the gas was absorbed almost completely by fuming sulphuric acid, and presumably consisted of unburnt acetylene. Thus the acetylene actually burnt amounted to 17·3-8, or 9·3 c.c. This only yielded 3 c.c. of carbon dioxide, whereas its complete oxidation would yield 18·6 c.c. It is clear, then, that the oxidation stops at some intermediate stage. Carbon monoxide is probably the chief product, and would, of course, escape condensation by liquid air.

*Benzene* under some conditions gave a just perceptible glow, too faint to be easily studied, though attended by a notable rise of temperature. On one occasion, when the glow was not observable, a white solid deposited on the glass. This exploded spontaneously with a loud report, breaking the apparatus, and leaving a deposit of carbon. It was probably the " ozobenzene " of Houzeau and Renard.*

*Methane*, *Ethylene* and the vapours of *Petrol*, *Ether* and *Alcohol* gave no glow.

*Napthalene* and *Camphor* also failed to give a glow, though energetically acted on. The former was carbonised by ozone. The latter was changed to a dull opaque white solid, easily distinguished from the glittering translucent surface of the original camphor.

*Cyanogen* gave a white glow, with a purple tinge near the nozzle, when the ozone issued into it. The spectrum shows the cyanogen bands at λ388 and λ387 very strongly, also those at λ359 and λ421.

In addition to these, the spectrum shows a band at λ431, which is possibly the hydrocarbon band 431·5, introduced in consequence of the gas having been collected over water; and another unidentified band at λ415, which may eventually prove of interest in connection with the spectra of comets.

In this case, as in all the spectroscopic part of the work, I am much indebted to the kind assistance of my colleagues, Prof. Fowler and Mr. Eagle. Prof. Fowler may perhaps pursue the work on the cyanogen spectrum.

Further experiments on other aspects of the subject are in progress.

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* " Comptes Rendus," 76, p. 572, 1873.

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